JOURNAL OF ALGEBRA **203,** 596–620 (1998)
ARTICLE NO. JA977338

On the Orbit Method for the Lie Algebra of Vector Fields on a Curve

Rosane Ushirobira

Department of Mathematics, Université de Poitiers, 40, Avenue du Recteur Pineau, Poitiers, 86022, France

Communicated by Peter Littelmann

Received December 11, 1996

Let $\mathfrak{g}$ be the Lie algebra of vector fields on an affine smooth curve Σ. Our goal is to establish an orbit method for $\mathfrak{g}$. Since $\mathfrak{g}$ is infinite-dimensional, we face some technical problems. Without having groups acting on $\mathfrak{g}$, we try nevertheless to define the notion of "orbits." So, we focus our attention to a subspace $\mathfrak{g}_f^*$ of $\mathfrak{g}^*$. This subspace consists of the "finite-dimensional orbits." To almost all λ in $\mathfrak{g}_f^*$ it corresponds a simple induced representation of $\mathfrak{g}$ whose annihilator is a primitive ideal. We conjecture that this ideal has a finite Gelfand–Kirillov codimension. What we are actually looking for is a bijection similar to Dixmier's bijection (in the finite-dimensional case) between the "orbits" of $\mathfrak{g}_f^*$ and certain primitive ideals of the enveloping algebra of $\mathfrak{g}$.

INTRODUCTION

Our goal in this paper is to propose an orbit method for the Lie algebra $\mathfrak{g}$ of vector fields $\mathfrak{g}$ on an affine smooth curve Σ. Using Dixmier's idea, we start by proving the simplicity of some induced representations of $\mathfrak{g}$.

For $\lambda \in \mathfrak{g}^*$, one can define the skew-symmetric bilinear form $d\lambda$ by $d\lambda(\eta, \xi) = \lambda[\eta, \xi]$, for all $\eta, \xi \in \mathfrak{g}$. Let $\mathfrak{g}_f^*$ be the set of finite rank elements of $\mathfrak{g}^*$, i.e., the elements $\lambda \in \mathfrak{g}^*$ satisfying $\operatorname{codim}_k \ker d\lambda < +\infty$. To each element of $\mathfrak{g}_f^*$, we associate a canonical polarization $\mathfrak{p}$. If $x \in \Sigma$ and $\lambda \in \mathfrak{g}_f^*$, define the order $\operatorname{ord}_x(\lambda)$ of λ in x so that $\operatorname{ord}_x(\lambda) = -1$ for almost all $x \in \Sigma$. The support $\operatorname{supp}(\lambda)$ of λ is the set of points $x \in \Sigma$ which satisfy $\operatorname{ord}_x(\lambda) \neq -1$. The main result in this paper is:

THEOREM. *Let $\lambda \in \mathfrak{g}_f^*$ and $\mathfrak{p}$ be its associated canonical polarization. Then*, $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *is a simple* $\mathfrak{g}$*-module if, and only if*, $\operatorname{ord}_x(\lambda) \neq 0$ *for all x in* $\operatorname{supp}(\lambda)$. *Nevertheless*, $\operatorname{Ann}_{\mathscr{U}(\mathfrak{g})} \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *is always a primitive ideal of* $\mathscr{U}(\mathfrak{g})$.

0021-8693/98 $25.00

It is this theorem that suggests the idea of a Dixmier map for $\mathfrak{g}$. More precisely, the fact that one can produce primitive ideals from whichever elements of $\mathfrak{g}^*_f$ should give rise to an orbit method.

In Section 2, we identify $\mathfrak{g}^*_f$ to the subspace $\mathfrak{g}^*_{fs}$ of elements of $\mathfrak{g}^*$ having finite support. For that, we show an interesting lemma about the dimension of the kernel of an element λ in $\mathfrak{g}^*$: it turns out that either $\dim_k \ker d\lambda$ is 0 or 1, or $\ker d\lambda$ is finite-codimensional and in this case, λ has finite support. We find in the Section 3, the canonical polarizations at elements of $\mathfrak{g}^*_f$. They are called canonical thanks to a flag of subalgebras of $\mathfrak{g}$. Naturally, this construction is inspired by the construction of Vergne polarization (for the finite-dimensional solvable Lie algebras). In Section 4, we prove the theorem stated above. Finally, in Section 5, we propose a Dixmier map for $\mathfrak{g}$.

1. NOTATIONS AND KNOWN GENERALITIES

(1) Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let Σ be a smooth irreducible affine curve. We denote by $\mathbf{A}$ the ring of regular functions over Σ. Recall that if Σ is embedded in $\mathbb{A}^n$ and defined by the equations $F_1 = \cdots = F_r = 0$ with $F_i \in \mathbf{k}[T_1, \ldots, T_n]$, then we have $\mathbf{A} = \mathbf{k}[\Sigma] = \mathbf{k}[T_1, \ldots, T_n]/\mathscr{I}(\Sigma)$. The ideal $\mathscr{I}(\Sigma)$ is the radical ideal which defines Σ with $\mathscr{I}(\Sigma) = (F_1, \ldots, F_r)$. Denote by $\mathbf{K}$ the quotient field of $\mathbf{A}$.

(2) A divisor D for the curve Σ is a formal sum of points of Σ with multiplicities. That is, $D = \sum_i m_i x_i$ where $x_i \in \Sigma$ and $m_i \in \mathbb{Z}$. The support of D is $\operatorname{supp}(D) = \{x_i \mid m_i \neq 0\}$. We call D effective if all $m_i \geq 0$. There exists a bijection between the effective divisors of Σ and the ideals of $\mathbf{A}$.

(3) Recall the Chinese Remainder Theorem: let $x_1, \ldots, x_s \in \Sigma$ and $u_1, \ldots, u_s$ be local parameters at $x_1, \ldots, x_s$, respectively. Let $\tilde{\mathscr{O}} = \bigcap_{i=1}^{r} \mathscr{O}_{x_i}$ where $\mathscr{O}_{x_i}$ denotes the local ring at x_i. The ring $\tilde{\mathscr{O}}$ has $(u_1), \ldots, (u_s)$ as prime ideals. Let $p_1, \ldots, p_s \in \mathbb{N}$. If $f_1, \ldots, f_s \in \tilde{\mathscr{O}}$, then there exists $h \in \tilde{\mathscr{O}}$ such that $h \equiv f_i \pmod{u_i^{p_i}}$, for all $i = 1, \ldots, s$.

(4) We set $\mathfrak{g} \overset{\text{def}}{=} \operatorname{Der}_{\mathbf{k}}(\mathbf{A}, \mathbf{A})$ to be the Lie algebra of $\mathbf{k}$-derivations of $\mathbf{A}$. This algebra is also called the Lie algebra of vector fields on Σ. The bracket is given by $[\eta, \xi] = \eta\xi - \xi\eta$, $\forall \eta, \xi \in \mathfrak{g}$. We state below some useful properties of $\mathfrak{g}$. The first three properties come from the fact that $\mathfrak{g}$ is a projective $\mathbf{A}$-module of rank 1.

(4a) For all ξ, η non-zero in $\mathfrak{g}$, there exist f, g non-zero in $\mathbf{A}$ such that $f\xi = g\eta$. One can deduce that if ξ, η are non-proportional over $\mathbf{k}$, then $[\xi, \eta] \neq 0$.

(4b) Let $\xi, \eta \in \mathfrak{g}$ and $\varphi \in \mathbf{A}$. Then $\eta(\varphi)\xi - \xi(\varphi)\eta = 0$.

(4c) Let $\mathbf{L}$ be an $\mathbf{A}$-submodule of $\mathfrak{g}$. Then there exists a unique ideal $\mathcal{I}$ of $\mathbf{A}$ such that $\mathbf{L} = \mathcal{I}\mathfrak{g}$. If $\mathbf{L}$ is a non-zero $\mathbf{A}$-submodule, then $\mathbf{L}$ has finite codimension in $\mathfrak{g}$.

(4d) Let $x \in \Sigma$ and u be a local parameter at x. Then there exists a neighborhood U of x such that the restriction of any $\delta \in \mathfrak{g}$ to $\mathbf{k}[U]$ can be written as $\delta = \delta(u)\mathrm{d}/\mathrm{d}u$.

(4e) (Jordan [J]) The Lie algebra $\mathfrak{g}$ is simple.

2. AN INTERESTING SUBSPACE OF $\mathfrak{g}^*$

Let $\mathfrak{g}^*$ denote the dual of the Lie algebra $\mathfrak{g}$. The group of automorphisms of Σ is small, often $\{1\}$. However, we can define "orbits" in $\mathfrak{g}$ (in the sense of coadjoint orbits, see Section 5) and we can talk about finite-dimensional "orbits." We are interested on these finite dimensional "orbits" of $\mathfrak{g}^*$ (that means, the orbits of elements $\lambda \in \mathfrak{g}^*$ such that $d\lambda$ has finite rank). In this section we will show that these "orbits" can be also characterized by the elements of $\mathfrak{g}^*$ whose support is a finite set of points of Σ. For this proof, we need an interesting result about the dimension of the kernel of $d\lambda$ for $\lambda \in \mathfrak{g}^*$.

DEFINITION 2.1. Let $x \in \Sigma$ and u be a local parameter at x. Let $m \in \mathbb{N}$. The element $\mathrm{T}_{x,m}$ of $\mathfrak{g}^*$ is defined as $\mathrm{T}_{x,m}(\delta) = (\mathrm{d}^m\delta(u)/\mathrm{d}u^m)(x)$ for δ written locally as $\delta(u)\mathrm{d}/\mathrm{d}u \in \mathfrak{g}$.

Let δ be in $\mathfrak{g}$ written locally as $f\,\mathrm{d}/\mathrm{d}u$ for some $f \in \mathbf{A}$. This operator $\mathrm{T}_{x,m}$ applied to δ gives the coefficient of u^m at the formal power series of f.

PROPOSITION 2.2. *Let $\mathcal{I}$ be an ideal of* $\mathbf{A}$ *and let $D = \sum_{i=1}^{s} m_i x_i$ be its associated divisor. Then* $\{\mathrm{T}_{x_i,l} \mid l = 0, 1, \dots, m_i - 1,\ i = 1, 2, \dots, s\}$ *is a basis of* $(\mathfrak{g}/\mathcal{I}\mathfrak{g})^*$.

Proof. It follows from the Chinese Remainder Theorem (1.3). ∎

DEFINITION 2.3. Define $\mathfrak{g}_f^*(x)$ as the vector space generated by $\mathrm{T}_{x,m}$, for all $m \in \mathbb{N}$.

Notice that $\mathrm{T}_{x,m}$ depends on the local parameter u, but the space $\mathfrak{g}_f^*(x)$ doesn't.

DEFINITION 2.4. We say that $\delta \in \mathfrak{g}$ vanishes $n+1$ times at x if $\mathrm{d}^i\delta(u)/\mathrm{d}u^i(x) = 0$ for all $i = 0, 1, \dots, n$ when δ is written locally as $\delta(u)\mathrm{d}/\mathrm{d}u$. Set $\mathfrak{g}_i^x = \{\delta \in \mathfrak{g} \mid \delta$ vanishes $i+1$ times at $x\}$. It is clear that we have the sequence $\mathfrak{g} = \mathfrak{g}_{-1}^x \supset \mathfrak{g}_0^x \supset \mathfrak{g}_1^x \supset \mathfrak{g}_2^x \supset \cdots$. To simplify the notation we will write sometimes $\mathfrak{g}_i$ instead of $\mathfrak{g}_i^x$.

We shall see now some properties of the subspaces $\mathfrak{g}_i$. Recall that for every $i \in \mathbb{N}$, we consider $u^i\, \mathrm{d}/\mathrm{d}u$ as an element of $\mathfrak{g}$.

LEMMA 2.5.

(i) *We have* $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. *In particular,* $\mathfrak{g}_i$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{2i+1}$.

(ii) *We have* $\dim_{\mathbf{k}} \underline{\mathfrak{g}_{i-1}/\mathfrak{g}_i} = 1$, *for all* $i = 0, 1, \ldots$ *and* $\{\overline{u^i\, \mathrm{d}/\mathrm{d}u}\}$ *generates* $\mathfrak{g}_{i-1}/\mathfrak{g}_i$ *where* $\overline{u^i\, \mathrm{d}/\mathrm{d}u}$ *denotes the class of* $u^i\, \mathrm{d}/\mathrm{d}u$ *in* $\mathfrak{g}_{i-1}/\mathfrak{g}_i$. *Then* $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}_i = i + 1$.

(iii) *The subspace* $\mathfrak{g}_i$ *is an* **A**-*submodule of* $\mathfrak{g}$. *In fact,* $\mathfrak{g}_i = \mathscr{P}_i \mathfrak{g}$ *where* $\mathscr{P}_i$ *is the ideal formed by the elements of* **A** *that vanishes* $i + 1$ *times at* x. *In other words,* $\mathscr{P}_i = \mathscr{M}_x^{i+1}$ *with* $\mathscr{M}_x = \{f \in \mathbf{A} \mid f(x) = 0\}$.

Proof. (i) Let $\delta = f\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}_i$ and $\eta = g\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}_j$. Then their formal series can be written as $f = \alpha_{i+1}u^{i+1} + \alpha_{i+2}u^{i+2} + \cdots$ with $m!\alpha_m = \mathrm{d}^m f/\mathrm{d}u^m(x)$ and $g = \beta_{j+1}u^{j+1} + \beta_{j+2}u^{j+2} + \cdots$ with $n!\beta_n = \mathrm{d}^n g/\mathrm{d}u^n(x)$. Thus $[\delta, \eta] = \sum_{k=i+j+2}^{\infty}(\sum_{r+s=k} s\alpha_r\beta_s - \sum_{p+q=k} p\alpha_p\beta_p)u^{k-1}\, \mathrm{d}/\mathrm{d}u$. Hence $[\delta, \eta] \in \mathfrak{g}_{i+j}$. In particular, if $i = j$ we have

$$[\delta, \eta] = \sum_{k=2i+3}^{\infty}\left(\sum_{r+s=k} s\alpha_r\beta_s - \sum_{p+q=k} p\alpha_p\beta_q\right)u^{k-1}\frac{\mathrm{d}}{\mathrm{d}u}.$$

That implies $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{2i+1} \subset \mathfrak{g}_i$.

(ii) It is clear that $u^i\, \mathrm{d}/\mathrm{d}u$ is not in $\mathfrak{g}_i$. Moreover, let $\delta = f\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}_{i-1}/\mathfrak{g}_i$. We write $f = \alpha_i u^i + \alpha_{i+1}u^{i+1} + \cdots$ with $\alpha_i \neq 0$. We obtain that $u^i\, \mathrm{d}/\mathrm{d}u$ generates $\mathfrak{g}_{i-1}/\mathfrak{g}_i$. As a consequence, $\dim_{\mathbf{k}} \mathfrak{g}_{i-1}/\mathfrak{g}_i = 1$.

(iii) Take $\delta \in \mathfrak{g}_i$. Then $\delta = \delta(u)\mathrm{d}/\mathrm{d}u$ and $\delta(u) = \alpha_{i+1}u^{i+1} + \alpha_{i+2}u^{i+2} + \cdots$ in the usual notation. Let $f \in \mathbf{A}$, $f = \beta_0 + \beta_1 u + \cdots$. This implies $f\delta(u) = \beta_0\alpha_{i+1}u^{i+1} + (\beta_0\alpha_{i+2} + \beta_1\alpha_{i+1})u^{i+2} + \cdots$ which is an element of $\mathfrak{g}_i$. It comes that $\mathfrak{g}_i$ is an **A**-submodule of $\mathfrak{g}$. ∎

Recall the notation of the enveloping algebra $\mathscr{U}(\mathfrak{g})$ of $\mathfrak{g}$. This algebra possesses a canonical filtration given by $\{\mathscr{U}_n(\mathfrak{g})\}_{n\geq 0}$ where $\mathscr{U}_n(\mathfrak{g})$ is the vector subspace of $\mathscr{U}(\mathfrak{g})$ generated by the products $\delta_1\delta_2\ldots\delta_p$ with $\delta_1, \delta_2, \ldots, \delta_p \in \mathfrak{g}$ and $p \leq n$.

LEMMA 2.6. *For every* $k \in \mathbb{N}$, *we have*:

(i) $[\mathfrak{g}_k, \mathfrak{g}] \subset \mathfrak{g}_{k-1}$

(ii) $\mathfrak{g}_k\mathscr{U}_n(\mathfrak{g}) \subset \mathscr{U}(\mathfrak{g})\mathfrak{g}_{k-n}$ *for all* $n \leq k$.

Proof. (i) It is easy to see from the lemma above that $[\mathfrak{g}_k, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{k-1}$.

(ii) We have $\mathfrak{g}_k\mathfrak{g} \subset \mathfrak{g}\mathfrak{g}_k + [\mathfrak{g}_k, \mathfrak{g}] \subset \mathfrak{g}\mathfrak{g}_k + \mathfrak{g}_{k-1} \subset \mathscr{U}(\mathfrak{g})\mathfrak{g}_{k-1}$. So by induction on n, this implies $\mathfrak{g}_k\mathscr{U}_n(\mathfrak{g}) = \mathfrak{g}_k\mathscr{U}_{n-1}(\mathfrak{g})\mathfrak{g} + \mathfrak{g}_k\mathscr{U}_{n-1}(\mathfrak{g}) \subset \mathscr{U}(\mathfrak{g})\mathfrak{g}_{k-n+1}\mathfrak{g} + \mathscr{U}(\mathfrak{g})\mathfrak{g}_{k-n+1} \subset \mathscr{U}(\mathfrak{g})\mathfrak{g}_{k-n}$. ∎

DEFINITION 2.7. Let $\lambda \in \mathfrak{g}_f^*(x)$. The *order* of λ at x is defined as being the smallest integer m such that $\lambda(\mathfrak{g}_m^x) = 0$. We denote it by $\mathrm{ord}_x(\lambda) = m$. It is clear that $\lambda(\mathfrak{g}_i^x) = 0$ for all $i \geq \mathrm{ord}_x(\lambda)$.

Remark 2.8. Let λ be an element of $\mathfrak{g}_f^*(x)$. Hence λ can be written uniquely as $\lambda = \sum_{i=0}^{s} \beta_i \mathrm{T}_{x,i}$ for a certain $s \in \mathbb{N}$ and $\beta_i \in \mathbf{k}$ with $\beta_s \neq 0$. Suppose $\mathrm{ord}_x(\lambda) = m$. It is clear that $s = m$. Let $\delta \in \mathfrak{g}_{m-1}$. We can easily see that $\lambda(\delta) = \beta_m \mathrm{T}_{x,m}(\delta)$ since $\mathrm{T}_{x,i}(\delta) = 0$, for all $i = 0, \dots, m-1$.

PROPOSITION 2.9. *Let $x \in \Sigma$. Let $\lambda \in \mathfrak{g}_f^*(x)$ of order m. Then,*

(i) *If m is even, one has $\mathfrak{g}^\lambda = \mathfrak{g}_{m+1}$.*

(ii) *If m is odd, one has $\mathfrak{g}_{m+1} \subset \mathfrak{g}^\lambda$ and $\dim_{\mathbf{k}} \mathfrak{g}^\lambda / \mathfrak{g}_{m+1} = 1$.*

Proof. Since $\mathrm{ord}_x(\lambda) = m$, one can write $\lambda = \sum_{i=0}^{m} \beta_i \mathrm{T}_{x,i}$ with $\beta_m \neq 0$. Let $\delta = f\,\mathrm{d}/\mathrm{d}u \in \mathfrak{g}^\lambda$. Write $f = \alpha_0 + \alpha_1 u + \alpha_2 u^2 + \cdots$. For every $n \in \mathbb{N}$ we have

$$\begin{aligned}
0 = d\lambda\left(\delta, u^n \frac{\mathrm{d}}{\mathrm{d}u}\right) &= \lambda\left(\left(nu^{n-1}f - u^n \frac{\mathrm{d}f}{\mathrm{d}u}\right)\frac{\mathrm{d}}{\mathrm{d}u}\right) \\
&= \lambda\left(\left(n\alpha_0 u^{n-1} + (n-1)\alpha_1 u^n + (n-2)\alpha_2 u^{n+1} + \cdots\right)\frac{\mathrm{d}}{\mathrm{d}u}\right) \\
&= \lambda\left(\left(\sum_{i=0}^{\infty} (n-i)\alpha_i u^{n+i-1}\right)\frac{\mathrm{d}}{\mathrm{d}u}\right) \\
&= \lambda\left(\left(\sum_{i=n-1}^{\infty} (2n-i-1)\alpha_{i-n+1} u^i\right)\frac{\mathrm{d}}{\mathrm{d}u}\right).
\end{aligned}$$

(i) Suppose m pair, $m = 2k$. If $n = 2k+1$, then $\lambda((\sum_{i=m}^{\infty}(4k - i + 1)\alpha_{i-2k}u^i)\mathrm{d}/\mathrm{d}u) = 0$. By Remark 2.8, we get $\beta_m \mathrm{T}_{x,m}((\sum_{i=m}^{\infty}(4k - i + 1)\alpha_{i-2k}u^i)\mathrm{d}/\mathrm{d}u) = 0$. Thus $\beta_m(m+1)\alpha_0 = 0$. One obtains $\alpha_0 = 0$.

By induction, suppose $\alpha_i = 0$, $\forall i = 0, \dots, j$ for $0 \leq j \leq 2k$. Put $n = 2k + 1 - (j+1)$. Then $\lambda((\sum_{i=m}^{\infty}(4k - 2j - i - 1)\alpha_{i-2k+j+1}u^i)\mathrm{d}/\mathrm{d}u) = 0$. So by Remark 2.8

$$\beta_m \mathrm{T}_{x,m}\left(\left(\sum_{i=m}^{\infty} (4k - 2j - i - 1)\alpha_{i-2k+j+1} u^i\right)\frac{\mathrm{d}}{\mathrm{d}u}\right) = 0.$$

Finally $\beta_m(2k - 2j - 1)\alpha_{j+1} = 0$. Since $2j \neq 2k - 1$, $\alpha_{j+1} = 0$. Then $\delta \in \mathfrak{g}_{m+1}$. That implies $\mathfrak{g}^\lambda \subset \mathfrak{g}_{m+1}$.

We obtain that $[\mathfrak{g}_{m+1}, \mathfrak{g}] \subset \mathfrak{g}_m$ using Lemma 2.6(i). It is easy to see that $\mathfrak{g}_{m+1} \subset \mathfrak{g}^\lambda$. Indeed, $\lambda[\mathfrak{g}_{m+1}, \mathfrak{g}] \subset \lambda(\mathfrak{g}_m) = 0$. As a consequence, we have $\mathfrak{g}^\lambda = \mathfrak{g}_{m+1}$.

(ii) Suppose m is odd, $m = 2k+1$. Proceeding as in the even case, we obtain also that $\mathfrak{g}_{m+1} \subset \mathfrak{g}^{\lambda}$. Let us show that $\dim_{\mathbf{k}} \mathfrak{g}^{\lambda}/\mathfrak{g}_{m+1} = 1$.

If $n = 2k+2$, then using the same reasoning as in the even case, one has $\alpha_0 = 0$. As previously, we suppose $\alpha_i = 0$, $\forall i = 0, 1, \dots, j$ for $0 \leq j \leq k-1$. Take $n = 2k+2-(j+1)$. We have $\beta_m \mathrm{T}_{x,m}((\sum_{i=m}^{\infty}(4k-2j-i+1)\alpha_{i-2k+j}u^i)\mathrm{d}/\mathrm{d}u) = 0$. So $\beta_m(2k-2j)\alpha_{j+1} = 0$. Since $j \neq k$, then $\alpha_{j+1} = 0$. Hence $\alpha_0 = \alpha_1 = \cdots = \alpha_k = 0$.

Suppose $n = k+1$. Then $d\lambda(\delta, u^n\,\mathrm{d}/\mathrm{d}u) = \lambda((\sum_{i=m}^{\infty}(2k+1-i)\alpha_{i-k}u^i)\mathrm{d}/\mathrm{d}u) = \beta_m(2k+1-m)\alpha_{k+1}$.

Therefore α_{k+1} can be any element of $\mathbf{k}$. Since $\alpha_0 = \alpha_1 = \cdots = \alpha_k = 0$, we have that $\lambda((\sum_{i=n+k}^{\infty}(2n-i-1)\alpha_{i-n+1}u^i)\mathrm{d}/\mathrm{d}u) = 0$.

If $n = k$, then $\lambda((\sum_{i=2k}^{\infty}(2k-i-1)\alpha_{i-k+1}u^i)\mathrm{d}/\mathrm{d}u) = 0$.

Recall that $\lambda = \beta_0\mathrm{T}_{x,0} + \cdots + \beta_{m-1}\mathrm{T}_{x,m-1} + \beta_m\mathrm{T}_{x,m}$. So $-(2k)!\alpha_{k+1} + (-2)(2k+1)!\alpha_{k+2} = 0$, i.e., $\alpha_{k+1} + 2(2k+1)\alpha_{k+2} = 0$.

Similarly when $n = k-i$ for $i = 1, \dots, k$: $(2k-i)!(i+1)\alpha_{k+1} + (2k-i+1)!(i+2)\alpha_{k+2} + \cdots + (2k+1)!(2i+2)\alpha_{k+i+2} = 0$.

Thus let us take $\eta = g\,\mathrm{d}/\mathrm{d}u \in \mathfrak{g}^{\lambda}$ such that $g = \alpha_{k+1}u^{k+1} + \alpha_{k+2}u^{k+2} + \cdots$ where $\alpha_i \in \mathbf{k}\,\alpha_{k+1}$ for every $i = k+1, \dots, m+1$. Choose $\alpha_{k+1} = 1$ in the series of g. We have then $g = u^{k+1} - (1/2m)u^{k+2} - (m - 3/8m^2)u^{k+3} + \cdots$. This element η does not belong to $\mathfrak{g}_{m+1}$. Moreover, it is easy to see that η generates $\mathfrak{g}^{\lambda}/\mathfrak{g}_{m+1}$. ∎

For $r \in \mathbb{R}$, $[r]$ denotes the integral part of r.

COROLLARY 2.10. *Let $x \in \Sigma$. Let $\lambda \in \mathfrak{g}_f^*(x)$ of order m. Then, the codimension of $\mathfrak{g}^{\lambda}$ is $m+2$ if m is even and $m+1$ if m is odd, i.e.,* $\operatorname{codim}_{\mathbf{k}} \mathfrak{g}^{\lambda} = 2[m/2] + 2$.

Proof. From Lemma 2.5(ii), we have $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}_{m+1} = m+2$. If m is even, then $\mathfrak{g}^{\lambda} = \mathfrak{g}_{m+1}$. Then, $\operatorname{codim}_{\mathbf{k}} \mathfrak{g}^{\lambda} = m+2$. Suppose m is odd. One obtains $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^{\lambda} = \dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}_{m+1} - \dim_{\mathbf{k}} \mathfrak{g}^{\lambda}/\mathfrak{g}_{m+1} = m+1$. ∎

2.1. *The Subspace $\mathfrak{g}_{fs}^*$.* Let $x \in \Sigma$ and u be a local parameter at x. Let $m \in \mathbb{N}$. We remark that the operators $\mathrm{T}_{x,m}$'s are linearly independent. Therefore we can form the direct sum of the $\mathfrak{g}_f^*(x)$'s. We define:

DEFINITION 2.11. The subspace $\mathfrak{g}_{fs}^*$ is defined by $\mathfrak{g}_{fs}^* = \oplus_{x \in \Sigma} \mathfrak{g}_f^*(x)$.

Remark 2.12. The subspace $\mathfrak{g}_{fs}^*$ consists of the elements of $\mathfrak{g}^*$ to which one can associate a support.

DEFINITION 2.13. Let $\lambda \in \mathfrak{g}_{fs}^*$. Therefore, there exists a unique way of writing λ as a finite sum $\sum_{\substack{x \in I \\ I \text{ finite}}} \lambda_x$ where $\lambda_x \in \mathfrak{g}_f^*(x)$. The *order* of λ at x is $\operatorname{ord}_x(\lambda) \overset{\text{def}}{=} \operatorname{ord}_x(\lambda_x)$.

Remark that $\operatorname{ord}_x 0 = -1$. In particular, $\operatorname{ord}_x(\lambda) = -1$ for almost all x in Σ.

DEFINITION 2.14. Let $\lambda \in \mathfrak{g}^*_{fs}$. Therefore λ is a finite sum $\sum_{\substack{x \in I \\ I \text{ finite}}} \lambda_x$ with $\lambda_x \in \mathfrak{g}^*_f(x)$. Define the *support* of λ as $\operatorname{supp}(\lambda) = \{x \in \Sigma \mid \operatorname{ord}_x(\lambda) \neq -1\}$.

We associate a divisor D of Σ to an element λ in $\mathfrak{g}^*_{fs}$ as follows:

DEFINITION 2.15. Let $\lambda \in \mathfrak{g}^*_{fs}$, $\lambda = \sum_{x \in \operatorname{supp}(\lambda)} \lambda_x$ with $\lambda_x \in \mathfrak{g}^*_f(x)$. The *associated divisor of* λ is defined as $D_\lambda = \sum_{x \in \operatorname{supp}(\lambda)} (\operatorname{ord}_x(\lambda) + 1)x$.

If D_λ is the associated divisor of λ, then $\operatorname{supp}(\lambda) = \operatorname{supp}(D_\lambda)$. Besides, $\mathfrak{g}^*_f(x)$ can be seen as $\mathfrak{g}^*_f(x) = \{\lambda \in \mathfrak{g}^*_f \mid \operatorname{supp}(\lambda) = \{x\}\}$.

EXAMPLE 2.1. Let $x \in \Sigma$ and $m \in \mathbb{N}$. Take $\mathrm{T}_{x,m} \in \mathfrak{g}^*_{fs}$. It is clear that $\operatorname{ord}_x(\mathrm{T}_{x,m}) = m$ and $\operatorname{supp}(\mathrm{T}_{x,m}) = \{x\}$. The associated divisor of $\mathrm{T}_{x,m}$ is $D_{\mathrm{T}_{x,m}} = (m+1)x$.

2.2. *The Subspace* $\mathfrak{g}^*_{fr}$. Let $\lambda \in \mathfrak{g}^*$. Recall we can define the skew-symmetric bilinear form $d\lambda\colon \mathfrak{g} \times \mathfrak{g} \to \mathbf{k}$ where $d\lambda(\eta, \xi) = \lambda[\eta, \xi]$, for all $\eta, \xi \in \mathfrak{g}$. We denote by $\mathfrak{g}^\lambda$ the kernel of $d\lambda$. That means $\mathfrak{g}^\lambda = \{\eta \in \mathfrak{g} \mid d\lambda(\eta, \xi) = 0, \forall \xi \in \mathfrak{g}\}$. Sometimes we use also the notation $\ker d\lambda$.

LEMMA 2.16. *Let* $\lambda \in \mathfrak{g}^*$. *Then*:

(i) *One has* $[\mathfrak{g}^\lambda, \mathfrak{g}] \subset \ker \lambda$.

(ii) *The subspace* $\mathfrak{g}^\lambda$ *is a Lie subalgebra of* $\mathfrak{g}$.

(iii) *If* $\mathscr{I}$ *is an ideal of* $\mathbf{A}$ *such that* $\mathscr{I}\mathfrak{g} \subset \ker \lambda$, *then* $\mathscr{I}^2 \mathfrak{g} \subset \mathfrak{g}^\lambda$.

Proof. The assertions (i) and (ii) are obvious. For (iii), for all $f, g \in \mathscr{I}$, $\xi, \eta \in \mathfrak{g}$, we have $[fg\xi, \eta] = fg[\xi, \eta] - (\eta(f)g + f\eta(g))\xi$. We deduce then $[\mathscr{I}^2\mathfrak{g}, \mathfrak{g}] \subset \mathscr{I}\mathfrak{g}$. Therefore $\lambda[\mathscr{I}^2\mathfrak{g}, \mathfrak{g}] \subset \lambda(\mathscr{I}\mathfrak{g}) = 0$. ∎

DEFINITION 2.17. We say that $d\lambda$ has *finite rank* if $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^\lambda < +\infty$.

DEFINITION 2.18. We define $\mathfrak{g}^*_{fr} = \{\lambda \in \mathfrak{g}^* \mid d\lambda \text{ has finite rank}\}$.

Remark 2.19. With the definition of this subspace $\mathfrak{g}^*_{fr}$, we focus our attention on the finite-dimensional "orbits" of $\mathfrak{g}^*$. This definition is inspired by the finite dimensional case of the orbit method. Indeed, if $\mathscr{A}$ is the algebraic adjoint group of a finite-dimensional Lie algebra $\mathfrak{g}$, then the tangent space to the coadjoint orbit $\mathscr{A} \cdot \lambda$ for $\lambda \in \mathfrak{g}^*$ is isomorphic to $\mathfrak{g}/\mathfrak{g}^\lambda$.

EXAMPLE 2.2. Consider $\Sigma = \mathbb{C} \setminus \{0\}$. Thus its affine coordinate ring is $\mathbf{A} = \mathbb{C}[T, T^{-1}]$, the Laurent polynomials algebra. In this case, we have that $\mathfrak{g} = \mathbf{W} \stackrel{\text{def}}{=} \oplus_{n \in \mathbb{Z}} \mathbb{C} L_n$ is the Witt algebra with $L_n = T^{n+1}\, \mathrm{d}/\mathrm{d}T$ for all $n \in \mathbb{Z}$.

With our definition of $\mathrm{T}_{x,m}$ ($x \in \Sigma, m \in \mathbb{N}$), we have $\mathrm{T}_{x,m}(f\, \mathrm{d}/\mathrm{d}T) = f^{(m)}(x)$, $\forall f \in \mathbf{A}$ where $f^{(m)}(x)$ denotes the mth derivative of f with respect to T evaluated at x. K. Bennani [Be, Proposition 10.3] showed that the rank of $d\mathrm{T}_{x,m}$ is $m+2$ if m is even and $m+1$ if m is odd. Hence $\mathrm{T}_{x,m} \in \mathfrak{g}^*_{fr}$.

PROPOSITION 2.20. Let $x \in \Sigma$ and $m \in \mathbb{N}$. Then, the rank of $d\mathrm{T}_{x,m}$ is $m+2$ if m is even and $m+1$ if m is odd.

Proof. Let u be a local parameter at x.

(i) Suppose m is even. Let $\delta = f\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}$ (written locally) and assume that f can be written as $f = \alpha_{m+2} u^{m+2} + \alpha_{m+3} u^{m+3} + \cdots$ with $i!\alpha_i = \mathrm{d}^i f/\mathrm{d}u^i(x)$, for all $i = m+2, m+3, \ldots$. Let $\eta = g\, \mathrm{d}/\mathrm{d}u$ be any element of $\mathfrak{g}$ and write $g = \beta_0 + \beta_1 u + \cdots$ for the formal power series of g. One has $[\delta, \eta] = (f\, \mathrm{d}g/\mathrm{d}u - g\, \mathrm{d}f/\mathrm{d}u)\mathrm{d}/\mathrm{d}u$. Then

$$\begin{aligned} d\mathrm{T}_{x,m}(\delta, \eta) &= \mathrm{T}_{x,m}[\delta, \eta] \\ &= \mathrm{T}_{x,m}\left(\sum_{k=m+2}^{\infty} \left(\sum_{i+j=k} j\alpha_i \beta_j - \sum_{p+q=k} p\alpha_p \beta_q \right) u^{k-1} \frac{\mathrm{d}}{\mathrm{d}u} \right). \end{aligned}$$

Since the coefficient of u^m is zero, the expression above is equal to zero. That shows that $\{f\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g} \mid \mathrm{d}^i f/\mathrm{d}u^i(x) = 0, \ \forall i = 0, \ldots, m+1\} \subset \ker d\mathrm{T}_{x,m}$.

Now, we shall prove the opposite inclusion. Let $\delta = f\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}$ with $f = \alpha_0 + \alpha_1 u + \cdots$ and let $\eta = g\, \mathrm{d}/\mathrm{d}u \in \mathfrak{g}$ as before. If $\delta \in \ker d\mathrm{T}_{x,m}$, we obtain

$$0 = d\mathrm{T}_{x,m}(\delta, \eta) = \mathrm{T}_{x,m}\left(\sum_{k=0}^{\infty} \left(\sum_{i+j=k} j\alpha_i \beta_j - \sum_{p+q=k} p\alpha_p \beta_q \right) u^{k-1} \frac{\mathrm{d}}{\mathrm{d}u} \right).$$

Since $\delta \in \ker d\mathrm{T}_{x,m}$, the coefficient of u^m in $[\delta, \eta]$ is equal to zero. Hence

$$\sum_{i+j=m+1} j\alpha_i \beta_j - \sum_{p+q=m+1} p\alpha_p \beta_q = 0. \tag{2.1}$$

Put $g = u^l$ for $l = 0, 1, \ldots, m+1$. From the expression (2.1) comes

$$l\alpha_{m-l+1} - (m-l+1)\alpha_{m-l+1} = 0, \qquad \text{i.e., } (m-(2l+1))\,\alpha_{m-l+1} = 0. \tag{2.2}$$

Since m is even, we get $\alpha_{m-l+1} = 0$, for all $l = 0, 1, \dots, m+1$.

(ii) When m is odd, the proof goes exactly the same as above, except for the two last lines. In this case, the expression (2.2) implies that $\alpha_{m-l+1} = 0$ when $l \neq m + 1/2$.

Then

$$\ker d\mathrm{T}_{x,m} = \begin{cases} \bigcap_{j=0}^{m+1} \ker \mathrm{T}_{x,j} & \text{if } m \text{ is even,} \\ \bigcap_{\substack{j=0 \\ j \neq m+1/2}}^{m+1} \ker \mathrm{T}_{x,j} & \text{if } m \text{ is odd.} \end{cases}$$

Henceforth the space $\mathscr{T}_{x,m}$ is a finite codimensional Lie subalgebra of $\mathfrak{g}$. ∎

Finally one obtains:

LEMMA 2.21. *We have* $\mathfrak{g}^*_{fs} \subset \mathfrak{g}^*_{fr}$.

Proof. Let $\lambda = \sum_{\substack{i \in I \\ I \text{ finite}}} \alpha_{x_i, m_i} \mathrm{T}_{x_i, m_i} \in \mathfrak{g}^*_{fs}$. Then $\bigcap_{i \in I} \ker d\mathrm{T}_{x_i, m_i} \subset \mathfrak{g}^{\lambda}$. From the previous proposition, $\ker d\mathrm{T}_{x_i, m_i}$ has finite codimension in $\mathfrak{g}$, for all $i \in I$. So, $\bigcap_{i \in I} \ker d\mathrm{T}_{x_i, m_i}$ has also finite codimension (recall that I is finite). It follows that $\mathfrak{g}^{\lambda}$ has finite codimension in $\mathfrak{g}$. ∎

To show that $\mathfrak{g}^*_{fr} \subset \mathfrak{g}^*_{fs}$, we need the useful lemma:

LEMMA 2.22. *Let* $\lambda \in \mathfrak{g}^*$ *such that* $d\lambda$ *is a degenerated bilinear form. Then*,

(1) *either* $\dim_{\mathbf{k}} \mathfrak{g}^{\lambda} = 1$,

(2) *or there exists a non-zero* **A**-*submodule* **M** *of* $\mathfrak{g}$ *contained in* $\mathfrak{g}^{\lambda}$. *In this case*, $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^{\lambda} \leq +\infty$.

Proof. Assume that $\dim_{\mathbf{k}} \mathfrak{g}^{\lambda} \geq 2$. We can find ξ, η two linearly independent elements in $\mathfrak{g}^{\lambda}$. Take φ in **A**. One has $\eta(\varphi)\xi - \xi(\varphi)\eta = 0$ by (1.4b). We compute $[\eta, \varphi\xi] = \eta(\varphi)\xi + \varphi[\eta, \xi]$ and $[\varphi\eta, \xi] = \varphi[\eta, \xi] - \xi(\varphi)\eta$. We add these two brackets and we get $[\eta, \varphi\xi] + [\varphi\eta, \xi] = (\eta(\varphi)\xi - \xi(\varphi)\eta) + 2\varphi[\eta, \xi] = 2\varphi[\eta, \xi]$. By Lemma 2.16(i), since $\varphi[\eta, \xi] \in \ker \lambda$, then $\mathbf{A}[\eta, \xi]$ is contained in $\ker \lambda$. Notice that $[\eta, \xi] \neq 0$ whereas the bracket of two elements in $\mathfrak{g}$ is zero if, and only if, they are proportional over **k** (see (1.4a)). Let $\mathbf{L} = \mathbf{A}[\eta, \xi]$. Therefore **L** is a non-zero **A**-submodule of $\mathfrak{g}$. Using (1.4c), one can find an ideal $\mathscr{I}$ of **A** such that $\mathbf{L} = \mathscr{I}\mathfrak{g}$. Since $\mathscr{I}\mathfrak{g} \subset \ker \lambda$, then $\mathscr{I}^2\mathfrak{g} \subset \ker d\lambda$ by Lemma 2.16(iii). But $\mathbf{M} = \mathscr{I}^2\mathfrak{g}$ is a finite codimensional **A**-submodule of $\mathfrak{g}$ by (1.4c). Hence $\mathfrak{g}^{\lambda}$ has finite codimension. ∎

EXAMPLE 2.3. In the previous lemma, one should take in account that the cases $\dim_{\mathbf{k}} \mathfrak{g}^\lambda = 1$ and $\dim_{\mathbf{k}} \mathfrak{g}^\lambda = 0$ are also possible as shown in this example. Consider the Witt algebra $\mathbf{W} = \oplus_{n\in\mathbb{Z}} \mathbb{C} L_n$. Let $\lambda \in \mathbf{W}^*$ be defined as:

(a) $\lambda(L_0) = 1$ and $\lambda(L_n) = 0$, when $n \neq 0$. So $\mathfrak{g}^\lambda = \mathbb{C}L_0$. Thus $\dim_{\mathbf{k}} \mathfrak{g}^\lambda = 1$.

(b) $\lambda(L_r) = 1$ and $\lambda(L_n) = 0$ for $n \neq r$ and $r \neq 0$. Then $\mathfrak{g}^\lambda = \mathbb{C}L_{r/2}$ if r is even and $\mathfrak{g}^\lambda = 0$ if r is odd. Therefore we have $\dim_{\mathbf{k}} \mathfrak{g}^\lambda = 1$ if r is even.

THEOREM 2.23. *Let* $\lambda \in \mathfrak{g}^*$. *Then*:

(1) *either* $\dim_{\mathbf{k}} \mathfrak{g}^\lambda \leq 1$,

(2) *or* $\lambda \in \mathfrak{g}^*_{fs}$. *In this case*, $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^\lambda < +\infty$.

Proof. Let λ be an element of $\mathfrak{g}^*$ such that $\dim_{\mathbf{k}} \mathfrak{g}^\lambda \geq 2$. Using Lemma 2.22, we can find a non-zero **A**-submodule **L** of $\mathfrak{g}$ such that $\mathbf{L} \subset \ker\lambda$. On the other hand, **L** can be written $\mathscr{I}\mathfrak{g}$ with $\mathscr{I}$ an ideal of **A** (by (1.4c)). Then, λ belongs to $(\mathfrak{g}/\mathscr{I}\mathfrak{g})^*$.

To every ideal $\mathscr{I}$ one can associate a divisor $D = \sum_{i=1}^s m_i x_i$ of Σ. The set $\{\mathrm{T}_{x_i,l} \mid l = 0,1,\dots,m_i-1,\ \ i = 1,2,\dots,s\}$ consists of a basis of $(\mathfrak{g}/\mathscr{I}\mathfrak{g})^*$. So, λ can be written as $\lambda = \sum_{i=1}^s \sum_{l=0}^{m_i-1} \alpha_{i,l} \mathrm{T}_{x_i,l}$ for some $\alpha_{i,l} \in \mathbf{k}$. Therefore λ belongs to $\mathfrak{g}^*_{fs}$. ∎

COROLLARY 2.24. *The subspaces* $\mathfrak{g}^*_{fs}$ *and* $\mathfrak{g}^*_{fr}$ *are equal.*

Proof. The inclusion $\mathfrak{g}^*_{fs} \subset \mathfrak{g}^*_{fr}$ was shown in Lemma 2.21. It is enough to apply the theorem above to obtain $\mathfrak{g}^*_{fr} \subset \mathfrak{g}^*_{fs}$. ∎

From now on, we shall denote $\mathfrak{g}^*_f \stackrel{\text{def}}{=} \mathfrak{g}^*_{fr} = \mathfrak{g}^*_{fs}$. This equality shows that in fact, the finite-dimensional "orbits" of $\mathfrak{g}^*$ come from the elements of $\mathfrak{g}^*$ determined by a finite number of points of Σ as mentioned at the beginning of this section.

PROPOSITION 2.25. *Let* $\lambda \in \mathfrak{g}^*_f$. *Write* $\lambda = \sum_{x\in \operatorname{supp}(\lambda)} \lambda_x$ *with* $\lambda_x \in \mathfrak{g}^*_f(x)$. *Suppose that* $\operatorname{supp}(\lambda) = \{x_1,\dots,x_s\}$. *Let* $m_i = \operatorname{ord}_{x_i}(\lambda)$, $i = 1,\dots,s$. *Then* $\mathfrak{g}^\lambda = \bigcap_{i=1}^s \mathfrak{g}^{\lambda_{x_i}}$. *In particular*, $\operatorname{codim}_{\mathbf{k}} \mathfrak{g}^\lambda = \sum_{i=1}^s 2q_i + 2$ *where* $q_i = [m_i/2]$.

Proof. It is clear that $\bigcap_{i=1}^s \mathfrak{g}^{\lambda_{x_i}} \subset \mathfrak{g}^\lambda$. Put $m_i = \operatorname{ord}_{x_i}(\lambda)$.

Let u_i be a local parameter at x_i. If $\delta \in \mathfrak{g}^\lambda$, then $d\lambda(\delta,\eta) = 0$ for all $\eta \in \mathfrak{g}$. Therefore $d\lambda(\delta, f\eta) = 0$ for all $f \in \mathbf{A}$, $\eta \in \mathfrak{g}$.

We take $j \in \{1,\dots,s\}$. By (1.3), we can find $h \in \tilde{\mathscr{O}} = \bigcap_{i=1}^s \mathscr{O}_{x_i}$ so that $h \equiv 1 \ (\mathrm{mod}(u_j^{m_j+1}))$ and $h \equiv 0 \ (\mathrm{mod}(u_i^{m_i+1}))$, $\forall i \neq j$. Thus, $0 = d\lambda(\delta, h\eta)$

$= m_j d\lambda_{x_j}(\delta, \eta)$. That means that $\delta \in \ker d\lambda_{x_j}$. It follows that $\mathfrak{g}^\lambda \subset \bigcap_{i=1}^s \mathfrak{g}^{\lambda_{x_i}}$.

Moreover, that implies that $\operatorname{codim}_{\mathbf{k}} \mathfrak{g}^\lambda = \sum_{i=1}^s \operatorname{codim}_{\mathbf{k}} \mathfrak{g}^{\lambda_{x_i}}$. Using Corollary 2.10, we get $\operatorname{codim}_{\mathbf{k}} \mathfrak{g}^\lambda = \sum_{i=1}^s 2([m_i/2] + 1)$. ∎

3. POLARIZATIONS OF $\mathfrak{g}$ AT AN ELEMENT OF $\mathfrak{g}_f^*$

As in the finite-dimensional Lie algebras case studied by Dixmier [D], the primitive ideals we are looking for rise from induced representations. These representations of $\mathfrak{g}$ come from polarizations. In this section, we recall the definition of Vergne polarization used by Dixmier. Then we construct similar polarizations for the Lie algebra $\mathfrak{g}$ of vector fields.

We start by considering $\mathfrak{g}$ an arbitrary Lie algebra over an algebraically closed field $\mathbf{k}$ with $\operatorname{char}(\mathbf{k}) = 0$. Let $\lambda \in \mathfrak{g}^*$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. The orthogonal subspace $\mathfrak{h}^\perp$ with respect to $d\lambda$ is defined by $\mathfrak{h}^\perp = \{\delta \in \mathfrak{g} \mid d\lambda(\delta, \eta) = 0, \forall \eta \in \mathfrak{h}\}$.

DEFINITION 3.1. Let $\lambda \in \mathfrak{g}^*$. A *polarization* of $\mathfrak{g}$ at λ is a maximal totally isotropic subalgebra with respect to $d\lambda$.

Let $\mathfrak{g}$ be a Lie algebra of dimension n. Let $\mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n$ be a flag of subspaces of $\mathfrak{g}$. Let $\lambda \in \mathfrak{g}^*$, $\lambda_i = \lambda|_{\mathfrak{g}_i}$, and $\mathfrak{p}_n = \mathfrak{g}_1^{\lambda_1} + \cdots + \mathfrak{g}_n^{\lambda_n}$. It is a trivial linear algebra result that $\mathfrak{p}_n$ is a maximal totally isotropic subspace of $\mathfrak{g}$.

PROPOSITION 3.2. *If $\mathfrak{g}_0 \lhd \mathfrak{g}_1 \lhd \cdots \lhd \mathfrak{g}_n$ is a flag of ideals of $\mathfrak{g}$, then $\mathfrak{p}_n$ is a Lie subalgebra of $\mathfrak{g}$ and moreover, $\mathfrak{p}_n$ is a polarization of $\mathfrak{g}$ at λ.*

Proof. See Dixmier [D, Chap. 1, Proposition 1.12.10]. ∎

This polarization is called the *Vergne polarization* associated to the flag $\mathscr{D} = (\mathfrak{g}_0, \dots, \mathfrak{g}_n)$. We denote it by $\mathfrak{p}(\lambda, \mathscr{D})$. In what follows we will see that for certain flags of subalgebras we also obtain a maximal totally isotropic subalgebra.

We are interested in the polarizations of our Lie algebra $\mathfrak{g}$ of vector fields at elements of $\mathfrak{g}_f^*$. Let $x \in \Sigma$ and u be a local parameter at x. Recall the definition of the subspaces $\mathfrak{g}_i$ of $\mathfrak{g}$ (2.4).

PROPOSITION 3.3. *Let $\lambda \in \mathfrak{g}_f^*(x)$ of order m. Then $\mathfrak{g}_{[m/2]}$ is a polarization of $\mathfrak{g}$ at λ.*

Proof. Put $\mathfrak{h} = \mathfrak{g}_q$. We will first prove that $\mathfrak{h}$ is totally isotropic, i.e., $\mathfrak{h} \subset \mathfrak{h}^\perp$. Let $\delta, \eta \in \mathfrak{h}$. From Lemma 2.5(i), it follows $[\delta, \eta] \in \mathfrak{g}_{2q+1}$. That means that $[\delta, \eta] \in \mathfrak{g}_{m+1}$ if m is even and $[\delta, \eta] \in \mathfrak{g}_m$ if m is odd. Thus, $\lambda([\delta, \eta]) = 0$.

Let us now show that $\mathfrak{h}^{\perp} \subset \mathfrak{h}$. Let $\delta = f\,\mathrm{d}/\mathrm{d}u \in \mathfrak{h}^{\perp}$ and $\eta = g\,\mathrm{d}/\mathrm{d}u \in \mathfrak{h}$. We write $f = \alpha_0 + \alpha_1 u + \cdots$ and $g = \beta_{q+1}u^{q+1} + \beta_{q+2}u^{q+2} + \cdots$. So, $[\delta, \eta] = \sum_{k=q+1}^{\infty}(\sum_{r+s=k} s\alpha_r\beta_s - \sum_{p+q=k} p\alpha_p\beta_q)u^{k-1}\,\mathrm{d}/\mathrm{d}u$. One knows that $\xi_l = u^l\,\mathrm{d}/\mathrm{d}u$ belongs to $\mathfrak{h}$, for all $l \geq q+2$. Hence, if we replace η by ξ_l and taking the coefficient of u^m in the formal power series of $[\delta, \xi_l](u)$, we obtain $l\alpha_{m+1-l} - (m+1-l)\alpha_{m+1-l} = 0$ (recall that $\lambda[\delta, \xi_l] = 0$). Thus $(2l - m - 1)\alpha_{m+1-l} = 0$, $\forall l \geq q+2$. If m is odd, then $m = 2q+1$. We have $l > q+1$, so $2l - m - 1 \neq 0$. Therefore $\alpha_i = 0$, $\forall i = 0, \dots, q$. If m is even, then $2l - m - 1 \neq 0$, for all l. One obtains that $\alpha_{m+1-l} = 0$, $\forall l \geq q+2$. So $\alpha_0 = \alpha_1 = \alpha_2 = \cdots = \alpha_q = 0$ and again, $\delta \in \mathfrak{h}$. ∎

Let $\lambda \in \mathfrak{g}_f^*$. Suppose $\mathrm{supp}(\lambda) = \{x_1, \dots, x_x\}$. Write $\lambda = \sum_{i=1}^{s}\lambda_i$ where $\lambda_i \in \mathfrak{g}_f^*(x_i)$ with $\mathrm{ord}_{x_i}(\lambda_i) = m_i$. Take $\mathfrak{p}_i = \mathfrak{g}_{q_i}$ with $q_i = [m_i/2]$ for all $i = 1, \dots, s$. From the previous proposition, the subalgebra $\mathfrak{p}_i$ is a polarization of $\mathfrak{g}$ at λ_i.

LEMMA 3.4. *The Lie subalgebra $\mathfrak{p} = \bigcap_{i=1}^{s}\mathfrak{p}_i$ is a polarization of $\mathfrak{g}$ at λ.*

Proof. First, let us prove that $\mathfrak{p} \subset \mathfrak{p}^{\perp}$. One recalls that $\mathfrak{p}_i = \mathfrak{p}_i^{\perp}$, $\forall i = 1, 2, \dots, s$ (by the proposition above) and then $\mathfrak{p} \subset \mathfrak{p}_i = \mathfrak{p}_i^{\perp} \subset \mathfrak{p}^{\perp}$.

For the opposite inclusion, notice that every $\mathfrak{p}_i$ is an **A**-submodule of $\mathfrak{g}$. Thus $\mathfrak{p}$ is also an **A**-submodule of $\mathfrak{g}$. Let $\mathscr{P}_{q_i} = \mathscr{M}_{x_i}^{q_i+1}$. Since $\mathfrak{p} = \bigcap_{i=1}^{s}\mathfrak{p}_i$, then using Lemma 2.2.5(iii) we get $\mathfrak{p} = (\mathscr{P}_{q_1} \cap \cdots \cap \mathscr{P}_{q_s})\mathfrak{g}$. The divisor associated to the ideal $(\mathscr{P}_{q_1} \cap \cdots \cap \mathscr{P}_{q_s})\mathfrak{g}$ is $\sum_{i=1}^{s}(q_i + 1)x_i$. Hence $\mathfrak{g}/\mathfrak{p}$ has dimension $\sum_{i=1}^{s}(q_i + 1)$ by Proposition 2.2.

One has $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{p}^{\perp} \leq \sum_{i=1}^{s}(q_i + 1)$ whereas $\mathfrak{p} \subset \mathfrak{p}^{\perp}$.

Since $\mathfrak{p} \cap \mathfrak{g}^{\lambda} \subset \mathfrak{g}^{\lambda}$, then $\dim_{\mathbf{k}} \mathfrak{g}/(\mathfrak{p} \cap \mathfrak{g}^{\lambda}) \geq \dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^{\lambda}$. However, $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{g}^{\lambda} = \sum_{i=1}^{s} 2(q_i + 1)$ by Proposition 2.25. We get that $\dim_{\mathbf{k}} \mathfrak{g}/(\mathfrak{p} \cap \mathfrak{g}^{\lambda}) \geq 2\sum_{i=1}^{s}(q_i + 1)$.

Finally we recall that $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{p}^{\perp} = \dim_{\mathbf{k}} \mathfrak{p}/(\mathfrak{p} \cap \mathfrak{g}^{\lambda})$. We use $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{p}^{\perp} = \dim_{\mathbf{k}} \mathfrak{g}/(\mathfrak{p} \cap \mathfrak{g}^{\lambda}) - \dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{p}$ to obtain $\dim_{\mathbf{k}} \mathfrak{g}/\mathfrak{p}^{\perp} = \sum_{i=1}^{s}(q_i + 1)$. ∎

Take $\lambda \in \mathfrak{g}_f^*$, $\lambda = \sum_{x \in \mathrm{supp}(\lambda)}\lambda_x$ with $\lambda_x \in \mathfrak{g}_f^*(x)$. Define the divisor E_λ of Σ as $\sum_{x \in \mathrm{supp}(\lambda)}([\mathrm{ord}_x(\lambda)/2] + 1)x$.

THEOREM 3.5. *The Lie subalgebra $\mathfrak{p} = I_{E_\lambda}\mathfrak{g}$ is a polarization of $\mathfrak{g}$ at λ where I_{E_λ} is the ideal of the divisor E_λ.*

Proof. The ideal I_{E_λ} associated to the divisor E_λ is $I_{E_\lambda} = \bigcap_{x \in \mathrm{supp}(\lambda)}\mathscr{P}_x$ with $\mathscr{P}_x = \mathscr{M}_x^{\mathrm{ord}_x(\lambda)+1}$. For every $x \in \mathrm{supp}(\lambda)$, we put $\mathfrak{p}_x = \mathfrak{g}_{[m_x/2]}$ where $m_x = \mathrm{ord}_x(\lambda)$. One knows that $\mathfrak{p}_x$ is a polarization of $\mathfrak{g}$ at λ_x by the previous proposition. So by using Lemma 3.4, one has $\mathfrak{p} = \bigcap_{x \in \mathrm{supp}(\lambda)}\mathfrak{p}_x$ is a polarization of $\mathfrak{g}$ at λ. But, $\mathfrak{p}_x = \mathscr{P}_x\mathfrak{g}$ by Lemma 2.5(iii). Then it is easy to deduce that $\mathfrak{p} = I_{E_\lambda}\mathfrak{g}$. ∎

From now on, we call this polarization of an element $\lambda \in \mathfrak{g}_f^*$, *the canonical polarization associated to* λ. One should notice that in general there are some other polarizations of $\mathfrak{g}$ at λ. An example can be found in Bennani [Be] for the Witt algebra W.

Let $x \in \Sigma$ and u be a local parameter at x. Let $\lambda \in \mathfrak{g}_f^*(x)$ and $\mathfrak{p} = \mathfrak{g}_q$ its associated canonical polarization. This polarization is called canonical since the flag of subalgebras $\mathfrak{g} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ is canonical. The construction of this canonical polarization is inspired by the finite-dimensional case as in Proposition 3.2 where the Vergne polarization was constructed. A big difference is that in our case, there is no flag of ideals since $\mathfrak{g}$ is simple. On the other hand, one can write the flag of subalgebras above. We remark that this flag is infinite and not a flag of ideals in contrast to Vergne's flag. Then the idea consists of passing the quotient by a suitable subalgebra $\mathfrak{g}_m$ in order to obtain a finite flag.

4. INDUCED REPRESENTATIONS OF $\mathfrak{g}$

It is in this section that we complete the construction of the primitive ideals mentioned in the Introduction. For that, we take representations of $\mathfrak{g}$ induced by the canonical polarizations seen in Section 3. We will show that these representations are almost always simple. Anyway, we obtain primitive ideals of $\mathscr{U}(\mathfrak{g})$ at the end.

Let $\mathfrak{g}$ be an arbitrary Lie algebra over an algebraically closed field $\mathbf{k}$ of zero characteristic. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let W be an $\mathfrak{h}$-module with $\rho\colon \mathfrak{h} \to \mathrm{End}(W)$ the corresponding representation. Denote the induced representation of $\mathfrak{g}$ by ρ as $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(W)$. Let $\lambda \in \mathfrak{g}^*$ and $\mathfrak{p}$ be a polarization of $\mathfrak{g}$ at λ. Then, λ defines a one-dimensional representation of $\mathfrak{p}$ in $\mathbf{k}_\lambda$, where $\mathbf{k}_\lambda$ is the $\mathfrak{p}$-module of dimension one so that $w \in \mathfrak{p}$ acts by multiplication by $\lambda(w)$. We denote the induced representation V of $\mathfrak{g}$ by $\mathbf{k}_\lambda$ as $\mathrm{V} = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$.

We state below some useful properties. Their proofs can be found, for example, in Dixmier [D, Chap. 5].

Property 4.1. Let $\mathfrak{h}$ be a finite-codimensional Lie subalgebra of $\mathfrak{g}$ and W be an $\mathfrak{h}$-module. Suppose $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{h}$. Let W be an $\mathfrak{h}$-module. Then the natural morphism $\Phi\colon \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(W) \to \mathrm{Ind}_{\mathfrak{h}/\mathfrak{n}}^{\mathfrak{g}/\mathfrak{n}}(W)$ is an isomorphism.

Property 4.2. Let $\mathfrak{h}$ and $\mathfrak{t}$ be Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{h}$. Let W be a $\mathfrak{t}$-module. Then $\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(W) \simeq \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{t}}^{\mathfrak{h}}(W))$.

Property 4.3. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be finite-codimensional Lie subalgebras of $\mathfrak{g}$. Denote $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{a}_i$. Suppose $\operatorname{codim}_{\mathbf{k}} \mathfrak{a} = \sum_{i=1}^{n} \operatorname{codim}_{\mathbf{k}} \mathfrak{a}_i$. Let A_i be $\mathfrak{a}_i$-modules, $i = 1, \ldots, n$. Then $\operatorname{Ind}_{\mathfrak{a}}^{\mathfrak{g}}(A_1 \otimes \cdots \otimes A_n) \simeq \operatorname{Ind}_{\mathfrak{a}_1}^{\mathfrak{g}}(A_1) \otimes \cdots \operatorname{Ind}_{\mathfrak{a}_n}^{\mathfrak{g}}(A_n)$.

Property 4.4. Let $\mathfrak{h}$ be a finite-codimensional Lie subalgebra of $\mathfrak{g}$. Let W be an $\mathfrak{h}$-module and U be a $\mathfrak{g}$-module. Then $\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(W \otimes U) \simeq \operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(W) \otimes U$.

Let us recall the following theorem whose proof can be found in Dixmier [D, Chap. 6, Theorem 6.1.1]:

THEOREM 4.5 (Dixmier). *Let $\mathfrak{g}$ be a finite-dimensional solvable Lie algebra and $\mathscr{D}$ be a flag of ideals of $\mathfrak{g}$. Let $\lambda \in \mathfrak{g}^*$ and $\mathfrak{p}(\lambda, \mathscr{D})$ be the Vergne polarization associated to λ. Then $\operatorname{Ind}_{\mathfrak{p}(\lambda, \mathscr{D})}^{\mathfrak{g}}(\lambda)$ is simple.*

Consider again $\mathfrak{g}$ as the Lie algebra of vector fields on Σ. Let $x \in \Sigma$. We want to prove the simplicity of the induced representation $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ when λ has support at one point x and when $\mathfrak{p}$ is the canonical polarization associated to λ by Proposition 3.3. The simplicity of this representation is fundamental in the proof of the simplicity of $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$, when λ has finite support.

Recall the decreasing sequence $\mathfrak{g} \supset \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$ constructed in Section 3. Let $k \in \mathbb{N}$. Then one has $\mathfrak{g}/\mathfrak{g}_k \supset \mathfrak{g}_0/\mathfrak{g}_k \supset \mathfrak{g}_1/\mathfrak{g}_k \supset \mathfrak{g}_2/\mathfrak{g}_k \supset \cdots \supset \mathfrak{g}_{k-1}/\mathfrak{g}_k$. Since $[\mathfrak{g}_i, \mathfrak{g}_0] \subset \mathfrak{g}_i$ (see Lemma 2.5(i)), then the subalgebra $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}_0$. Hence, $\mathfrak{g}_0/\mathfrak{g}_k \triangleright \mathfrak{g}_1/\mathfrak{g}_k \triangleright \mathfrak{g}_2/\mathfrak{g}_k \triangleright \cdots \triangleright \mathfrak{g}_{k-1}/\mathfrak{g}_k$. We conclude that $\mathfrak{g}_0/\mathfrak{g}_k$ is a finite-dimensional solvable Lie algebra (since $\dim_{\mathbf{k}} \mathfrak{g}_{i-1}/\mathfrak{g}_i = 1$ (by Lemma 2.5(ii))).

LEMMA 4.6. *Let $\lambda \in \mathfrak{g}_f^*(x)$ of order m. Let $\mathfrak{p} = \mathfrak{g}_q$ be its associated polarization where $q = [m/2]$. Let $\lambda' = \lambda|_{\mathfrak{g}_0/\mathfrak{g}_{m+1}}$. Then, $\mathfrak{p}$ is in $\mathfrak{g}_0$ and $\mathfrak{p}/\mathfrak{g}_{m+1}$ is the Vergne polarization associated to λ'.*

Proof. Write $\lambda = \sum_{i=1}^{m} \beta_i T_{x,i}$. It is clear that for all i, $\mathfrak{g}_{i+1} \subset \ker dT_{x,i}$ (by Proposition 2.20). So, $\mathfrak{g}_{m+1} \subset \ker dT_{x,i}$, for all $i = 1, \ldots, m$. One deduces $\mathfrak{g}_{m+1} \subset \mathfrak{g}^{\lambda}$.

Let $\mathfrak{a}_i = \mathfrak{g}_{m+1-i}/\mathfrak{g}_{m+1}$, $\forall i = 0, 1, \ldots, m+1$. Put $\lambda_i = \lambda'|_{\mathfrak{a}_i}$, $\forall i = 0, 1, \ldots, m+1$. Take $\mathfrak{q}_{m+1} = \mathfrak{a}_1^{\lambda_1} + \cdots + \mathfrak{a}_{m+1}^{\lambda_{m+1}}$. Apply Proposition 3.2 to the flag $\mathscr{D} = \mathfrak{a}_0 \triangleleft \cdots \triangleleft \mathfrak{a}_{m+1}$ and to λ'. We obtain that $\mathfrak{p}(\lambda, \mathscr{D}) = \mathfrak{q}_{m+1}$ is a Vergne polarization associated to λ'. Now we remark that $\mathfrak{q}_{m+1} = \mathfrak{p}/\mathfrak{g}_{m+1}$. Indeed, $\mathfrak{p}/\mathfrak{g}_{m+1} = \mathfrak{a}_{m+1-q} \subset \mathfrak{a}_{m+1}$. Since $\mathfrak{p} = \mathfrak{p}^{\perp}$, then $\mathfrak{a}_{m+1-q} = \mathfrak{a}_{m+1-q}^{\lambda_{m+1-q}}$. But $\mathfrak{a}_{m+1-q}^{\lambda_{m+1-q}} \subset \mathfrak{q}_{m+1}$. Hence $\mathfrak{a}_{m+1-q} \subset \mathfrak{q}_{m+1}$. On the other hand, if $\delta \in \mathfrak{q}_{m+1}$, then $d\lambda(\delta, \mathfrak{q}_{m+1}) = 0$. That implies that $d\lambda'(\delta, \mathfrak{p}/\mathfrak{g}_{m+1}) = 0$. Thus $\delta \in \mathfrak{p}/\mathfrak{g}_{m+1}$. ∎

PROPOSITION 4.7. *Let $\lambda \in \mathfrak{g}_f^*(x)$ of order m. Let $\mathfrak{p}$ be its associated canonical polarization. Then* $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\lambda)$ *is simple.*

Proof. Using the above lemma, we know that $\mathfrak{p}/\mathfrak{g}_{m+1}$ is a Vergne polarization associated to $\lambda' = \lambda \mid_{\mathfrak{g}_0/\mathfrak{g}_{m+1}}$. We use the simplicity theorem for the finite-dimensional solvable case (4.5) and it results that $\operatorname{Ind}_{\mathfrak{p}/\mathfrak{g}_{m+1}}^{\mathfrak{g}_0/\mathfrak{g}_{m+1}}(\lambda) = \operatorname{Ind}_{\mathfrak{g}_{m+1}}^{\mathfrak{a}_{m+1}}(\lambda')$ is simple. It is enough now to use Property 4.1 and thus $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\lambda) \simeq \operatorname{Ind}_{\mathfrak{p}/\mathfrak{g}_{m+1}}^{\mathfrak{g}_0/\mathfrak{g}_{m+1}}(\lambda)$. ∎

We know that $\mathfrak{g}_0$ has codimension one in $\mathfrak{g}$. A basis of $\mathfrak{g}/\mathfrak{g}_0$ is $\{\tilde{e}\}$ (by Lemma 2.5(ii)), where $\tilde{e}$ denotes the class of $\mathrm{d}/\mathrm{d}u$ in $\mathfrak{g}/\mathfrak{g}_0$. We denote also $e = \mathrm{d}/\mathrm{d}u$. From now on, consider α as the character of $\mathfrak{g}_0$ so that $\alpha(u\,\mathrm{d}/\mathrm{d}u) = 1$ (i.e., $\alpha = \mathrm{T}_{x,1}$).

LEMMA 4.8. *Let W be a $\mathfrak{g}_0$-module with ρ: $\mathfrak{g}_0 \to \operatorname{End}(W)$. Let $L = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} W$ and $\{L_n\}_{n \in \mathbb{N}}$ be the canonical filtration of L. Then $[z,e] + \alpha(z)e$ belongs to $\mathfrak{g}_0$ and $z \cdot (e^n \otimes w) = e^n \otimes (\rho - n\alpha)(z)w + e^{n-1} \otimes (n\rho([z,e] + \alpha(z)e) - \binom{n}{2}\alpha[z,e])w + L_{n-2}$ for all $z \in \mathfrak{g}_0$, $w \in W$, and $n \in \mathbb{N}$.*

Proof. Let $z \in \mathfrak{g}_0$. For all $n \in \mathbb{N}$, we have the formula (see Dixmier [D, Chap. 2, Sect. 2]) $ze^n = \sum_{i=0}^n \binom{n}{i} e^{n-i}[z, \overbrace{e, \ldots, e}^{i}]$ where $[z, \overbrace{e, \ldots, e}^{i}] = [[[z,e],e],\ldots,e]$. It follows from this expression that $ze^n = e^n z + ne^{n-1}[z,e] + \binom{n}{2}e^{n-2}[z,e,e] + \mathscr{U}_{n-3}(\mathfrak{g})$. Therefore $ze^n = e^n z + ne^{n-1}([z,e] + \alpha(z)e) - n\alpha(z)e^n + \binom{n}{2}e^{n-2}([z,e,e] + \alpha([z,e])e) - \binom{n}{2}\alpha([z,e])e^{n-1} + \mathscr{U}_{n-2}(\mathfrak{g})$. But $[z,e] + \alpha(z)e$ belongs to $\mathfrak{g}_0$. Indeed, $\alpha([z,e]) = -\alpha(z)e$. Also $[z,e,e] + \alpha([z,e])e$ belongs to $\mathfrak{g}_0$. That implies that for all $w \in W$, $z \cdot (e^n \otimes w) = e^n \otimes (\rho - n\alpha)(z)w + e^{n-1} \otimes (n\rho([z,e] + \alpha(z)e) - \binom{n}{2}\alpha[z,e])w + L_{n-2}$. ∎

PROPOSITION 4.9. *Let W be a simple $\mathfrak{g}_0$-module. Let $L = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} W$ and $\{L_n\}_{n \in \mathbb{N}}$ be its canonical filtration. Let M be a non-trivial $\mathfrak{g}$-submodule of L. Then, there exists $n \in \mathbb{N}$ such that $L \simeq L_n \oplus M$ as $\mathfrak{g}_0$-modules.*

Proof. We have a $\mathfrak{g}$-module homomorphism $M \hookrightarrow L \xrightarrow{\varphi} N$ with $N \simeq L/M$. Put $\varphi(L_n) = N_n$, $\forall n$. Denote ρ: $\mathfrak{g}_0 \to \operatorname{End}(W)$. Thanks to Lemma 4.8, we obtain a $\mathfrak{g}_0$-module homomorphism $\mathbf{k}\tilde{e}^{n+1} \otimes W \simeq \mathbf{k}_{\rho-(n+1)\alpha} \otimes W$. Indeed, one can verify that $z \cdot (\tilde{e}^{n+1} \otimes w) = \tilde{e}^{n+1} \otimes (\rho - (n+1)\alpha)(z)w$, for all $z \in \mathfrak{g}_0$. So $L_{n+1}/L_n \simeq \mathbf{k}\tilde{e}^{n+1} \otimes W \simeq \mathbf{k}_{\rho-(n+1)\alpha} \otimes W$. One concludes that L_{n+1}/L_n is a simple $\mathfrak{g}_0$-module since W is a simple $\mathfrak{g}_0$-module. Take n minimum such that $L_{n+1} \cap M \neq 0$. There exists a natural inclusion f: $(M \cap L_{n+1}) \hookrightarrow L_{n+1}/L_n$. But L_{n+1}/L_n is simple. Henceforth f is an isomorphism. From that it results $N_{n+1} = L_{n+1}/(M \cap L_{n+1}) \simeq L_n$.

As a consequence, using the fact that M is a $\mathfrak{g}$-module, we conclude that $L_{n+2}/(M \cap L_{n+2})$ is isomorphic to a $\mathfrak{g}_0$-submodule of L_n. By induction, we have that $L_{n+s}/(M \cap L_{n+s})$ is also isomorphic to a $\mathfrak{g}_0$-submodule of L_n, $\forall s \geq 0$. Finally, we have the following exact sequence that splits,

$$O \to M \to L \underset{\psi}{\overset{\varphi}{\rightleftarrows}} N \to 0,$$

where $\psi \circ \varphi = \mathrm{Id}$ and $\psi(y+m) = y+m$, $\forall y \in L_{n+s}/(M \cap L_{n+s}) \subset L_n$, $\forall m \in M$. Indeed, $\varphi \circ \psi(y+m) = \varphi(y+m) = y+m$ and $\psi \circ \varphi(y) = \psi(z+m) = z+m = y$. Therefore, $L \simeq L_n \oplus M$. ∎

The following lemma is obvious:

LEMMA 4.10. *Let M and N be $\mathfrak{g}_0$-modules. Let μ be a character of $\mathfrak{g}_0$. Then*, $\mathrm{Ext}^1(M \otimes \mathbf{k}_\mu, N \otimes \mathbf{k}_\mu) \simeq \mathrm{Ext}^1(M, N)$.

LEMMA 4.11. *Let W be a simple $\mathfrak{g}_0$-module with $\rho\colon \mathfrak{g}_0 \to \mathrm{End}(W)$. Let $A(z) = \rho([z,e] + \alpha(z)e)$ and $b(z) = \alpha([z,e])$, for all $z \in g_0$. Put $B = b \otimes \mathrm{Id}_W$. Then,*

(1) *b is a 1-cocycle with values in $\mathrm{Hom}(\mathbf{k}\tilde{e}, \mathbf{k})$ and B is a 1-cocycle with values in $\mathrm{Hom}(W \otimes \mathbf{k}\tilde{e}, W)$.*

(2) *A is a 1-cocycle with values in $\mathrm{Hom}(W \otimes \mathbf{k}\tilde{e}, W)$.*

Proof. For all $z, v \in \mathfrak{g}_0$,

$$\begin{aligned}[][[z,v],e] = {} & [[z,e] + \alpha(z)e, v] + \alpha(z)([v,e] + \alpha(v)e) \\ & + [z,[v,e] + \alpha(v)e] - \alpha(v)([z,e] + \alpha(z)e). \quad (4.1)\end{aligned}$$

Remark that $\alpha[\mathfrak{g}_0, \mathfrak{g}_0] = 0$ and $[z,e] + \alpha(z)e$ and $[v,e] + \alpha(v)e$ are in $\mathfrak{g}_0$.

(1) Let us show that b is a 1-cocycle with values in $\mathrm{Hom}(\mathbf{k}\tilde{e}, \mathbf{k})$. We need to show that $b[z,v] = \alpha(z)b(v) - \alpha(v)b(z)$, i.e., $\alpha[[z,v],e] = \alpha(z)\alpha[v,e] - \alpha(v)\alpha[z,e]$ for all $z, v \in \mathfrak{g}_0$. That comes from the expression (4.1) (recall $\alpha[\mathfrak{g}_0, \mathfrak{g}_0] = 0$). In particular, it also follows that $B = b \otimes \mathrm{Id}_W$ is a 1-cocycle with values in $\mathrm{Hom}(W \otimes \mathbf{k}\tilde{e}, W)$.

(2) Put $\rho_1\colon \mathfrak{g}_0 \to \mathrm{End}(W \otimes \mathbf{k}e)$ with $\rho_1 = \rho - \alpha$. We need to prove that $A[z,v] = \rho \circ A(v) - A(v) \circ \rho_1(z) - \rho(v) \circ A(z) + A(z) \circ \rho_1(v)$ for all $z, v \in \mathfrak{g}_0$. It is enough to show that $A[z,v] = [\rho(z), \rho([v,e] - \alpha(v)e)] + \alpha(z)\rho([v,e]\alpha(v)e) - [\rho(v), \rho([z,e] + \alpha(z)e)] - \alpha(v)\rho([z,e] + \alpha(z)e)$. Just use the formula (4.1). ∎

LEMMA 4.12. *Using the notations of Lemma* 4.11, *put for all* $n \in \mathbb{N}$, $\xi_n(z) = nA(z) - \binom{n}{2}B(z)$ *for* $z \in \mathfrak{g}_0$. *Take* $L = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$. *Then, the extension in* $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}^{n+1}, W \otimes \mathbf{k}\tilde{e}^n)$ *associated to* ξ_n *is* $0 \to L_n/L_{n-1} \to L_{n+1}/L_{n-1} \to L_{n+1}/L_n \to 0$.

Proof. Using Lemma 4.11, we can see that $[A]$ and $[B]$ are in $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$. So, $[\xi_n] \in \mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$. Consider the exact sequence

$$0 \to W \xrightarrow{\varphi} W \oplus (W \otimes \mathbf{k}\tilde{e}) \xrightarrow{\psi} W \otimes \mathbf{k}\tilde{e} \to 0, \tag{4.2}$$

where $\varphi(w) = (w, 0)$ and $\psi(w, w' \otimes \tilde{e}) = w' \otimes \tilde{e}$. Since $[\xi_n]$ is a cocycle in $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$, then we have a $\mathfrak{g}_0$-module structure on $W \oplus (W \otimes \mathbf{k}\tilde{e})$ given by $z \cdot (w, w' \otimes \tilde{e}) = (\rho(z)w + \xi_n(z)(w' \otimes \tilde{e}), \rho_1(z)(w' \otimes \tilde{e}))$.

That implies that the exact sequence (4.2) is the sequence associated to $[\xi_n]$ in $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$. However, $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}^{n+1}, W \otimes \mathbf{k}\tilde{e}^n) \simeq \mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$ by Lemma 4.10. Besides, $L_n/L_{n-1} \simeq W \otimes \mathbf{k}\tilde{e}^n$, $L_{n+1}/L_{n-1} \simeq W \otimes \mathbf{k}\tilde{e}^n \oplus W \otimes \mathbf{k}\tilde{e}^{n+1}$ and $L_{n+1}/L_n \simeq W \otimes \mathbf{k}\tilde{e}^{n+1}$. So we conclude that the sequence $0 \to L_n/L_{n-1} \to L_{n+1}/L_{n-1} \to L_{n+1}/L_n \to 0$ is associated to the cocycle $[\xi_n] \in \mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}^{n+1}, W \otimes \mathbf{k}\tilde{e}^n)$. ∎

PROPOSITION 4.13. *Let* W *be a simple* $\mathfrak{g}_0$*-module with* $\rho\colon \mathfrak{g}_0 \to \mathrm{End}(W)$. *Suppose* $\mathfrak{g}/\mathfrak{g}_0 \otimes W \simeq W$ *as* $\mathfrak{g}_0$*-modules. Then, either* $L = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} W$ *is a simple* $\mathfrak{g}$*-module or* W *can be extended to a* $\mathfrak{g}$*-module.*

Proof. If L is not simple, then from Proposition 4.9, one can find a $\mathfrak{g}$-submodule M of L and $r \in \mathbb{N}$ such that $L \simeq L_r \oplus M$. Recall the formula (see Lemma 4.8)

$$z \cdot (e^n \otimes w) = e^n \otimes (\rho - n\alpha)(z)w + e^{n-1} \otimes \left(n\rho([z, e] + \alpha(z)e) - \binom{n}{2}\alpha[z, e] \right) w + L_{n-2}.$$

Put $A(z) = \rho([z, e] + \alpha(z)e)$ and $B(z) = \alpha([z, e])$. We put also $\xi_n(z) = nA(z) - \binom{n}{2}B(z)$. Thus using Lemma 4.12, we have $[\xi_n] \in \mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W) \simeq \mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}^{n+1}, W \otimes \mathbf{k}\tilde{e}^n)$.

The exact sequence $0 \to L_r/L_{r-1} \to L_{r+1}/L_{r-1} \to L_{r+1}/L_r \to 0$ splits since $L \simeq L_r \oplus M$. We showed in Lemma 4.12 that this sequence is associated to the cocycle $[\xi_r]$. Then $[\xi_r] = 0$ in $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$.

Since one has $M \simeq \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W \otimes (\mathfrak{g}/\mathfrak{g}_0)^{\otimes(n+1)})$ by Proposition 4.9, then $M \simeq L$. One can write then $M \simeq M_r \oplus N$ for a certain $\mathfrak{g}$-module N of M where $\{M_n\}_{n \geq 0}$ is the canonical filtration of M. That implies that the exact sequence below splits:

$$0 \to L_{2r}/L_{2r-1} \to L_{2r+1}/L_{2r-1} \to L_{2r+1}/L_{2r} \to 0.$$

This sequence is associated to the cocycle $[\xi_{2r}]$. Hence $[\xi_{2r}] = 0$ in $\mathrm{Ext}^1(W \otimes \mathbf{k}\tilde{e}, W)$. We have then $[\xi_r] = [\xi_{2r}] = 0$, and thus $[\xi_r] = 0$. But $[\xi_r] = r[A] - \binom{r}{2}[B]$, therefore $[A] = [B] = 0$.

From $[\rho([z,e] + \alpha(z)e)] = 0$, it follows that there exists $\theta \in \mathrm{Hom}(W \otimes \mathbf{k}\tilde{e}, W)$ such that $\rho([z,e] + \alpha(z)e) = \rho(z) \circ \theta - \theta \circ \rho_1(z)$ where $\rho_1\colon \mathfrak{g}_0 \to \mathrm{End}(W \otimes \mathbf{k}\tilde{e})$ is given by $\rho_1(z) = (\rho - \alpha)(z)$. Thus $\rho([z,e] + \alpha(z)e) = [\rho(z), \theta] + \alpha(z)\theta$.

Define $\pi\colon \mathfrak{g} \to \mathrm{End}(W)$ so that $\pi|_{\mathfrak{g}_0} = \rho$ and $\pi(e) = \theta$. We compute $\pi[z,e] = \rho([z,e] + \alpha(z)e) - \alpha(z)\theta = [\rho(z), \theta]$. That means that the $\mathfrak{g}_0$-module W can be extended to a $\mathfrak{g}$-module. ∎

Take $\lambda \in \mathfrak{g}_f^*(x)$. Let $\mathfrak{p}$ be its associated canonical polarization. Let $W = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\lambda)$ where $\rho\colon \mathfrak{g}_0 \to \mathrm{End}(W)$ denotes the representation of $\mathfrak{g}_0$ corresponding to W. We proved in Proposition 4.7 that W is a simple $\mathfrak{g}_0$-module. Let $L = \mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W) = \oplus_{n \in \mathbb{N}} e^n \otimes W$. Let $L_n = \oplus_{r \le n} e^r \otimes W$ be given by the canonical filtration of L.

We need the results below in order to prove the simplicity of L.

PROPOSITION 4.14. *Suppose $\mathfrak{p} = \mathfrak{g}_0$ and $\lambda|_{\mathfrak{g}_0} \neq 0$. Then for all $n \in \mathbb{N}$ and $v \in L_n$, there exists $\tilde{u} \in \mathfrak{g}_0$ such that $\tilde{u} \cdot v \in L_0$ and $\tilde{u} \cdot v \neq 0$.*

Proof. Let w be the generator of W as a $\mathscr{U}(\mathfrak{g}_0)$-module. We take $\tilde{u} = u\,\mathrm{d}/\mathrm{d}u \in \mathfrak{g}_0$. One has $\tilde{u} \cdot w = \lambda(\tilde{u})w$. Since $\lambda|_{\mathfrak{g}_0} \neq 0$, then $\tilde{u} \cdot w \neq 0$.

For $n \ge 1$, we take $\tilde{u} = u^{n+1}\,\mathrm{d}/\mathrm{d}u \in \mathfrak{g}_n$. Let $v = \sum_{i=0}^n \alpha_i e^i \otimes w \in L_n$ with $\alpha_i \in \mathbf{k}$ and $\alpha_n \neq 0$. We know that

$$\Bigl[\tilde{u}, \overbrace{e, \dots, e}^{j \text{ times}}\Bigr] = (-1)^j (n+1) \cdots (n-j+2) u^{n-j+1} \frac{\mathrm{d}}{\mathrm{d}u}.$$

Observe that $\mathrm{ord}_x(\lambda) = 1$ whereas $\mathfrak{p} = \mathfrak{g}_0$ and $\lambda|_{\mathfrak{g}_0} \neq 0$. Hence $\lambda(\mathfrak{g}_i) = 0$ for $i \ge 1$. That implies

$$\tilde{u} \cdot v = \alpha_n \Bigl[\tilde{u}, \overbrace{e, \dots, e}^{n \text{ times}}\Bigr] \otimes w = 1 \otimes (-1)^n (n+1)!\,\alpha_n \lambda\Bigl(u \frac{\mathrm{d}}{\mathrm{d}u}\Bigr) w \in L_0.$$

Moreover, it is clear that $\tilde{u} \cdot v \neq 0$ since $\alpha_n \neq 0$ and $\lambda|_{\mathfrak{g}_0} \neq 0$. Then $\tilde{u} \cdot w \neq 0$. ∎

PROPOSITION 4.15. *Suppose $\mathfrak{p} = \mathfrak{g}_0$. Then $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is simple if, and only if, $\mathrm{ord}_x(\lambda) \neq 0$.*

Proof. Assume $\operatorname{ord}_x(\lambda) \neq 0$. Thus $\operatorname{ord}_x(\lambda) = 1$ and $\lambda|_{\mathfrak{g}_0} \neq 0$. Write $W = \mathscr{U}(\mathfrak{g}_0)\mathbf{k}_\lambda$. Suppose that there is M a $\mathfrak{g}$-submodule of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$. Let $v \in M$, $v \neq 0$. Then $v \in L_n$ for some $n \in \mathbb{N}$. By the previous proposition, there exists $\tilde{u} \in \mathfrak{g}_0$ so that $\tilde{u} \cdot v \in W$ and $\tilde{u}v \neq 0$. Hence, $M \cap W \neq 0$. Therefore $M \cap W = W$ since W is simple. Consequently, $\mathscr{U}(\mathfrak{g})W \subset M$. That implies $L \subset M$.

If $\operatorname{ord}_x(\lambda) = 0$ then $\lambda|_{\mathfrak{g}_0} = 0$, and thus $\oplus_{n \geq 1} e^n \otimes W$ is a $\mathfrak{g}$-submodule of $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$. ∎

PROPOSITION 4.16. *The polarization $\mathfrak{p}$ is contained in $\mathfrak{g}_1$ if, and only if, $\mathfrak{g}/\mathfrak{g}_0 \otimes W \simeq W$ (as $\mathfrak{g}_0$-modules).*

Proof. Suppose $\mathfrak{p} \subset \mathfrak{g}_1$. Then, $\alpha(\mathfrak{p}) = 0$ (recall α denotes $\mathrm{T}_{x,1}$). Therefore $\mathbf{k}_\lambda \simeq \mathbf{k}_{\lambda+\alpha}$ as $\mathfrak{p}$-modules. By Property 4.4, it results that $\mathbf{k}_\alpha \otimes \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\mathbf{k}_\lambda) \simeq \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\mathbf{k}_\alpha \otimes \mathbf{k}_\lambda)$. Finally, since $\mathbf{k}_\alpha \simeq \mathbf{k}\tilde{e}$, one has $\mathbf{k}\tilde{e} \otimes W \simeq \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\mathbf{k}_{\alpha+\lambda}) \simeq W$.

Conversely, assume that $\mathfrak{p} \not\subseteq \mathfrak{g}_1$. Hence $\mathfrak{p} = \mathfrak{g}_0$. Then W is equal to $\mathbf{k}_\lambda$, the one dimensional $\mathscr{U}(\mathfrak{g}_0)$-module. It is clear that $\mathbf{k}_\lambda \not\simeq \mathbf{k}_{\lambda+\alpha}$ as $\mathfrak{g}_0$-modules. Henceforth $\mathbf{k}e \otimes W \simeq \mathscr{U}(\mathfrak{g}_0)\mathbf{k}_{\lambda+\alpha} \not\simeq \mathscr{U}(\mathfrak{g}_0)\mathbf{k}_\lambda = W$. ∎

Finally we have:

THEOREM 4.17. *Let $\lambda \in \mathfrak{g}_f^*(x)$. Then the representation $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is simple if, and only if,* $\operatorname{ord}_x(\lambda) > 0$.

Proof. Suppose $\operatorname{ord}_x(\lambda) \geq 1$. Then $\mathfrak{p} \subset \mathfrak{g}_1$ or $\mathfrak{p} = \mathfrak{g}_0$.

If $\mathfrak{p} \subseteq \mathfrak{g}_1$, then by Proposition 4.16 it follows that $\mathfrak{g}/\mathfrak{g}_0 \otimes W \simeq W$ (as $\mathfrak{g}_0$-modules).

So either $L = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is simple or W possesses a $\mathfrak{g}$-module structure that extends its $\mathfrak{g}_0$-module structure (by Proposition 4.13). Suppose that L is not simple. Let $\pi: \mathfrak{g} \to \operatorname{End}(W)$ so that $\pi|_{\mathfrak{g}_0} = \rho$ where $\rho: \mathfrak{g}_0 \to \operatorname{End}(W)$ denotes the representation of $\mathfrak{g}_0$ corresponding to W. Let $m = \operatorname{ord}_x(\lambda)$. We have $\ker \rho \neq 0$ since $\mathfrak{g}_m \subset \ker \rho$. Indeed, recall that $\mathfrak{g}_m \subset \ker \lambda$ and that $z \in \mathfrak{g}_m \subset \mathfrak{p}$ acts on W by multiplication by $\lambda(z)$. Therefore, $\ker \rho = \mathfrak{g}_0$ since W is simple. Hence $\mathfrak{g}_0 \subset \ker \pi$. That implies $\ker \pi \neq 0$. Since $\mathfrak{g}$ is a simple Lie algebra by (1.4e), we get a contradiction. Thus $L = \operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is simple.

When $\mathfrak{p} = \mathfrak{g}_0$, the $\mathfrak{g}$-module $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is simple as shown in Proposition 4.15.

Suppose $\operatorname{ord}_x(\lambda) \leq 0$. If $\operatorname{ord}_x(\lambda) = -1$, then $\lambda \equiv 0$. This case is not interesting. If $\operatorname{ord}_x(\lambda) = 0$, then $\mathfrak{p} = \mathfrak{g}_0$ and from Proposition 4.15, we conclude that $\operatorname{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(W)$ is not simple. ∎

Finally one has:

THEOREM 4.18. *Let* $\lambda \in \mathfrak{g}_f^*(x)$ *and* $\mathfrak{p}$ *be its associated canonical polarization. Then the representation* $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *is simple if, and only if,* $\mathrm{ord}_x(\lambda) \neq 0$.

Proof. Consider $\mathrm{ord}_x(\lambda) \neq 0$. Then from the previous theorem we obtain the simplicity of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_0}(\lambda))$. The theorem follows then from Property 4.2.

The case $\mathrm{ord}_x(\lambda) = 0$ is also a direct consequence of Theorem 4.17. ∎

We want to prove now the simplicity of the induced representations by elements of $\mathfrak{g}_f^*$. Recall the following proposition that describes the simplicity of a $\mathfrak{g}$-module (Jacobson Density Theorem). The proof of this proposition can be found for example in Dixmier [D, Chap. 2, Sect. 6].

PROPOSITION 4.19. *The following assertions are equivalent*:

(i) *V is a simple $\mathfrak{g}$-module.*

(ii) *For all $x_1, \ldots, x_n$ linearly independent and $y_1, \ldots, y_n \in V$, there exists $u \in \mathscr{U}(\mathfrak{g})$ such that $ux_i = y_i$.*

(iii) *For all E_1 and E_2 finite-dimensional subspaces of V, one has a bijection* $I_{E_1, E_2} \hookrightarrow \mathrm{End}(E_1, E_2)$ *where* $I_{E_1, E_2} = \{u \in \mathscr{U}(\mathfrak{g}) \mid uE_1 \subset E_2\}$.

The results below will be used to prove the simplicity.

LEMMA 4.20. *Let $x \in \Sigma$ and $\lambda \in \mathfrak{g}_f^*(x)$. Let $\mathfrak{p}$ be its associated canonical polarization. Let $V = \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ and v be a generator of $\mathbf{k}_\lambda$. Then, there exists $k \in \mathbb{N}$ such that $\mathfrak{g}_k v = 0$.*

Proof. Let $m = \mathrm{ord}_x(\lambda)$. It is easy to see that $\mathfrak{g}_m v = 0$ since $\mathfrak{g}_m \subset \ker \lambda$ and $\mathfrak{g}_m$ is contained in $\mathfrak{p}$ (an element z acts on v by multiplication by $\lambda(z)$). ∎

PROPOSITION 4.21. *With the same notation as in Lemma* 4.20, *let E be a finite-dimensional subspace of V. Then, there exists $k \in \mathbb{N}$ such that $\mathfrak{g}_k \cdot z = 0$, for all $z \in E$.*

Proof. One has $E \subset \mathscr{U}_n(\mathfrak{g})v$ for some n. By Lemma 2.6(ii), for all $z \in E$, one has $\mathfrak{g}_{m+n} \cdot z \subset \mathfrak{g}_{m+n}\mathscr{U}_n(\mathfrak{g})v \subset \mathscr{U}(\mathfrak{g})\mathfrak{g}_m v$ where $m = \mathrm{ord}_x(\lambda)$. Hence $\mathfrak{g}_{m+n} \cdot z = 0$ by the above lemma. ∎

PROPOSITION 4.22. *With the same notation as in Lemma* 4.20, *let $E_1, \ldots, E_s$ be finite-dimensional subspaces of V. Let $z \in \mathfrak{g}$. Then, there exists $y \in \mathfrak{g}$ such that $y \cdot e_1 = z \cdot e_1$, for all $e_1 \in E_1$ and $y \cdot e_i = 0$, for all $e_i \in E_i$ whenever $i \neq 1$.*

Proof. By the previous proposition, we can find $k_1, \ldots k_s$ in $\mathbb{N}$ for $E_1, \ldots, E_s$. Then it is enough to find $y - z \in \mathfrak{g}_{k_1}$ and $y \in \mathfrak{g}_{k_i}$ *for* i $\neq 1$.

Write $z = f\,\mathrm{d}/\mathrm{d}u$. By (1.3), there exists $h \in \tilde{\mathscr{O}} = \bigcap_{i=1}^{r} \mathscr{O}_{x_i}$ such that $h \equiv f \pmod{u_1^{k_1}}$ and $h \equiv 0 \pmod{u_i^{k_i}}$, $i \neq 1$, where $u_1, \ldots, u_s$ are local parameters at $x_1, \ldots, x_s$, respectively. We take then $y = h\,\mathrm{d}/\mathrm{d}u$. ∎

PROPOSITION 4.23. *With the same notation as in Lemma* 4.20, *let* $E_1, \ldots, E_s$ *be finite-dimensional subspaces of* V. *Let* $z \in \mathscr{U}_m(\mathfrak{g})$. *Then, there exists* $\tilde{z} \in \mathscr{U}_m(\mathfrak{g})$ *such that* $\tilde{z} \cdot (e_1 \otimes \cdots \otimes e_s) = (z \cdot e_1) \otimes e_2 \otimes \cdots \otimes e_s$, *for all* $e_1, \ldots, e_s \in E_1, \ldots, E_s$.

Proof. Suppose $m = 1$. By the previous proposition, we can find $y \in \mathfrak{g}$ so that $y \cdot e_1 = z \cdot e_1$, $\forall e_1 \in E_1$ and $y \cdot e_1 = 0$, $\forall e_i \in E_i$ for $i \neq 1$. Therefore, we have $y \cdot (e_1 \otimes \cdots \otimes e_s) = \sum_{i=1}^{s} e_1 \otimes \cdots \otimes (y \cdot e_i) \otimes \cdots \otimes e_s = (y \cdot e_1) \otimes \cdots \otimes e_s$.

For an arbitrary m, we use induction. Let $z \in \mathscr{U}_m(\mathfrak{g})$. We write $z = \sum z_i$ where each z_i is of the form $z_i = y_i v_i$, with $v_i \in \mathscr{U}_{m-1}(\mathfrak{g})$ and $y_i \in \mathfrak{g}$. By induction, $\tilde{v}_i$ belongs to $\mathscr{U}_{m-1}(\mathfrak{g})$ such that $\tilde{v}_i \cdot e_1 = v_i \cdot e_i$, $\forall e_1 \in E_1$ and $\tilde{v}_i \cdot e_i = 0$, $\forall e_i \in E_i$ for $i \neq 1$. By (1.3), there exists $\tilde{y}_i \in \mathfrak{g}$ so that $\tilde{y}_i v_i \cdot e_1 = y_i v_i \cdot e_1$, $\forall e_1 \in E_1$ and $\tilde{y}_i \cdot e_i = 0$, $\forall e_i \in E_i$, $i \neq 1$.

We take $\tilde{z}_i = \tilde{y}_i \tilde{v}_i \in \mathscr{U}_m(\mathfrak{g})$. Put $\tilde{z} = \sum \tilde{z}_i$ and the result follows. ∎

THEOREM 4.24. *Let* $x_1, \ldots, x_s \in \Sigma$. *Let* $V_i = \mathrm{Ind}_{\mathfrak{p}_i}^{\mathfrak{g}}(\lambda_i)$ *where* $\lambda_i \in \mathfrak{g}_f^*(x_i)$ *and* $\mathfrak{p}_i$ *is the canonical polarization associated to* λ_i, $i = 1, \ldots, s$. *Suppose* $\mathrm{ord}_{x_i}(\lambda_i) \neq 0$, $i = 1, \ldots, s$. *Then* $V_1 \otimes \cdots \otimes V_s$ *is a simple* $\mathfrak{g}$-*module*.

Proof. Let $E = E_1 \otimes \cdots \otimes E_s$ and $F = F_1 \otimes \cdots \otimes F_s$ be finite-dimensional subspaces of $V_1 \otimes \cdots \otimes V_s$, where E_i, F_i are subspaces of V_i, $i = 1, \ldots, s$.

Let $\Phi \in \mathrm{End}(E_1 \otimes \cdots \otimes E_s, F_1 \otimes \cdots \otimes F_s) \simeq \mathrm{End}(E_1, F_1) \otimes \cdots \otimes \mathrm{End}(E_s, F_s)$. Therefore one can write $\Phi = \varphi_1 \otimes \cdots \otimes \varphi_s$ with $\varphi_i \in \mathrm{End}(E_i, F_i)$, $i = 1, \ldots, s$.

To show the simplicity of $V_1 \otimes \cdots \otimes V_s$, we need to find $a \in \mathscr{U}(\mathfrak{g})$ such that $a \cdot (e_1 \otimes \cdots \otimes e_s) = \Phi(e_1 \otimes \cdots \otimes e_s)$ for all $e_1, \ldots, e_s \in E_1, \ldots, E_s$ (by Proposition 4.19).

Using the simplicity of induced representations proved before (see Theorem 4.18) and by Proposition 4.19, we find $z_i \in \mathscr{U}(\mathfrak{g})$ so that $z_i|_{E_i} \subset F_i$, $i = 1, \ldots, s$ and moreover, $\varphi_i(y) = z_i \cdot y$, $\forall y \in E_i$, $i = 1, \ldots, s$. Since $z_i \in \mathscr{U}_{n_i}(\mathfrak{g})$ for some $n_i \in \mathbb{N}$, we write then $z_i = \sum_{j=0}^{n_i} z_{i,j}$ where $z_{i,j} \in \mathscr{U}_j(\mathfrak{g}) \setminus \mathscr{U}_{j-1}(\mathfrak{g})$.

For every $i = 1, \ldots, s$, we want to find $\tilde{z}_i \in \mathscr{U}(\mathfrak{g})$ so that $\tilde{z}_i \cdot (e_1 \otimes \cdots \otimes e_s) = (1 \otimes \cdots \otimes 1 \otimes z_i \otimes 1 \otimes \cdots \otimes 1)(e_1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_s)$ for all $e_1, \ldots, e_s \in E_1, \ldots, E_s$. That results directly from Proposition 4.23.

So, let us put $a = \tilde{z}_1 \cdots \tilde{z}_s \in \mathscr{U}(\mathfrak{g})$. We have for all $e_1, \ldots, e_s \in E_1, \ldots, E_s$,

$$\begin{aligned} a \cdot (e_1 \otimes \cdots \otimes e_s) &= \tilde{z}_1 \cdots \tilde{z}_s \cdot (e_1 \otimes \cdots \otimes e_s) \\ &= \tilde{z}_1 \cdots \tilde{z}_{s-1} \cdot (e_1 \otimes \cdots \otimes e_{s-1} \otimes z_s \cdot e_s) \\ &= z_1 \cdot e_1 \otimes \cdots \otimes z_s \cdot e_s \\ &= (\varphi_1 \otimes \cdots \otimes \varphi_s)(e_1 \otimes \cdots \otimes e_s) = \Phi(e_1 \otimes \cdots \otimes e_s). \end{aligned}$$

Thus, $V_1 \otimes \cdots \otimes V_s$ is a simple $\mathfrak{g}$-module. ∎

THEOREM 4.25. *Let $\lambda = \sum_{x \in \operatorname{supp}(\lambda)} \lambda_x \in \mathfrak{g}_f^*$ where $\lambda_x \in \mathfrak{g}_f^*(x)$ and let $\mathfrak{p}$ be the canonical polarization associated to λ. Then, $V = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a simple $\mathfrak{g}$-module if, and only if, $\operatorname{ord}_x(\lambda) \neq 0$ for every $x \in \operatorname{supp}(\lambda)$.*

Proof. Let $\operatorname{supp}(\lambda) = \{x_1, \ldots, x_n\}$. Put $V_i = \operatorname{Ind}_{\mathfrak{p}_{x_i}}^{\mathfrak{g}}(\lambda_{x_i})$, for all $i = 1, \ldots, s$.

Suppose $\operatorname{ord}_x(\lambda) \neq 0$ for all $x \in \operatorname{supp}(\lambda)$. We have $\mathfrak{p} = \bigcap_{x \in \operatorname{supp}(\lambda)} \mathfrak{p}_x$ (by Theorem 3.5) where $\mathfrak{p}_x$ is the canonical polarization associated to λ_x. So by Property 4.3, we obtain $V \simeq V_1 \otimes \cdots \otimes V_s$. It is enough now to apply the previous theorem.

If $\operatorname{ord}_x(\lambda) = 0$ for some $x \in \operatorname{supp}(\lambda)$, then by Theorem 4.18, one of the V_i's is not simple. Then $V \simeq V_1 \otimes \cdots \otimes V_s$ is not simple. ∎

5. A DIXMIER MAP

5.1. *The Orbits of* $\mathfrak{g}_f^*$. In order to define the orbits of $\mathfrak{g}_f^*$ without having the action of a group, we are going to define a equivalence relation in this space. Naturally, one is inspired by the finite-dimensional case.

Let $\lambda \in \mathfrak{g}^*$. Let $\mathfrak{m}_\lambda$ be the prime ideal which defines λ in $\mathscr{S}(\mathfrak{g})$. That is, $\mathfrak{m}_\lambda = \{u \in \mathscr{S}(\mathfrak{g}) \mid \lambda(u) = 0\}$. Recall that the Lie algebra $\mathfrak{g}$ acts on $\mathscr{S}(\mathfrak{g})$ by the adjoint action. Let $\mathscr{I}_\lambda$ be the biggest $\mathfrak{g}$-invariant prime ideal contained in $\mathfrak{m}_\lambda$. The existence of this ideal is given by Lemma 3.3.2 in Dixmier [D, Chap. 3].

We define the relation $\sim$ on $\mathfrak{g}_f^*$: for all λ, μ in $\mathfrak{g}_f^*$, $\lambda \sim \mu \overset{\text{def}}{\Leftrightarrow} \mathscr{I}_\lambda = \mathscr{I}_\mu$. It is clear that this relation is an equivalence relation.

5.2. *A Dixmier Map.* First, we show that one can associate a primitive ideal of $\mathscr{U}(\mathfrak{g})$ to any element of $\mathfrak{g}_f^*$ by taking the annihilator of the induced representations constructed in Section 4.

THEOREM 5.1. *For every $\lambda \in \mathfrak{g}_f^*$, $\operatorname{Ann}_{\mathscr{U}(\mathfrak{g})} \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a primitive ideal of $\mathscr{U}(\mathfrak{g})$.*

Proof. If $\operatorname{ord}_x(\lambda) \neq 0$ for all $x \in \operatorname{supp}(\lambda)$, then Theorem 4.25 says that $V = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is a simple $\mathfrak{g}$-module. Therefore $\operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}V$ is obviously a primitive ideal.

Write $\lambda = \sum_{i=1}^{n}\lambda_i$ with $\lambda_i \in \mathfrak{g}_f^*(x_i)$ where $\operatorname{supp}(\lambda) = \{x_1, \dots, x_n\}$. Now, put $\operatorname{ord}_{x_i}(\lambda) = m_i$ and suppose $m_i = 0$ for $i = 1, \dots, s$ $(s \leq n)$. So the canonical polarization associated to λ is $\mathfrak{p} = \mathfrak{g}_0^{x_1} \cap \cdots \cap \mathfrak{g}_0^{x_s} \cap \mathfrak{g}_{[m_{s+1}/2]}^{x_{s+1}} \cap \cdots \cap \mathfrak{g}_{[m_n/2]}^{x_n}$. Consider $\mu_i = \lambda_i + \alpha_i$ where α_i is the character of $\mathfrak{g}_0^{x_i}$. We can define then the following exact sequence as $\mathfrak{g}$-modules $0 \to \operatorname{Ind}_{\mathfrak{p}_i}^{\mathfrak{g}}(\mu_i) \to \operatorname{Ind}_{\mathfrak{p}_i}^{\mathfrak{g}}(\lambda_i) \to \mathbf{k} \to 0$ where we consider $\mathbf{k}$ as a trivial $\mathfrak{g}$-module. Denote $V_i = \operatorname{Ind}_{\mathfrak{p}_i}^{\mathfrak{g}}(\lambda_i)$ and $U_i = \operatorname{Ind}_{\mathfrak{p}_i}^{\mathfrak{g}}(\mu_i)$. Also denote $I_i = \operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}V_i$ and $J_i = \operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}U_i$. It follows from the exact sequence that J_i contains I_i. Now define $P_i = \mathscr{U}^+(\mathfrak{g})$ (the augmentation ideal) where $\mathscr{U}^+(\mathfrak{g})$ denotes $\oplus_{n \geq 1}\mathscr{U}_n(\mathfrak{g})$. We have that $P_iV_i \subset U_i$. That implies that $J_iP_iV_i = 0$. Therefore $J_iP_i \subset I_i$. The ideal I_i is completely prime (see Remark 5.4), hence either $P_i \subset I_i$ or $J_i \subset I_i$. But P_i has codimension one in $\mathscr{U}(\mathfrak{g})$ while I_i has infinite codimension (since $\{e_i^n, n \in \mathbb{N}\}$ is not in I_i where e_i denotes the element in $\mathfrak{g}$ but not in $\mathfrak{p}_i$). Therefore $J_i \subset I_i$. We conclude that $I_i = J_i$.

To get the final result about λ, one needs to take the tensor products $V \simeq \otimes V_i \hookrightarrow U \simeq \otimes U_i$ where $U = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mu)$. We then obtain $\operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mu) = \operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. ∎

Let us now define a map $\mathbf{I}$: $\mathfrak{g}_f^*/\sim \to \operatorname{Prim}\mathscr{U}(\mathfrak{g})$ by $\mathbf{I}(\{\lambda\}) = \operatorname{Ann}_{\mathscr{U}(\mathfrak{g})}\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ where $\{\lambda\}$ denotes the equivalence class of $\lambda \in \mathfrak{g}_f^*$. Let $\mathfrak{p}$ be the canonical polarization associated to λ.

Conjecture 1. The map $\mathbf{I}$ is well defined.

5.3. *About the Gelfand–Kirillov Dimension*

Let $\lambda \in \mathfrak{g}_f^*$. Let $\mathfrak{p}$ be its associated canonical polarization. Set $L = \operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. Let $\mathfrak{b}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{b}$. Let $\sigma: \mathscr{S}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$ be the symmetrization. We have a bijection $\sigma \otimes 1: \mathscr{S}(\mathfrak{b}) \otimes \mathbf{k}_\lambda \to \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} \mathbf{k}_\lambda$. Then we can write $L \simeq \mathscr{S}(\mathfrak{b}) \otimes \mathbf{k}_\lambda$.

PROPOSITION 5.2. *The Gelfand–Kirillov dimension of* $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *is equal to the dimension of* $\mathfrak{g}/\mathfrak{p}$.

Proof. Set $\dim_{\mathbf{k}} \mathfrak{b} = n$. We have $\mathscr{S}(\mathfrak{b}) \otimes \mathbf{k}_\lambda \simeq \mathbf{k}[X_1, \dots, X_n]$. The polynomial ring $\mathbf{k}[X_1, \dots, X_n]$ has Gelfand–Kirillov dimension n. ∎

Recall that for an arbitrary Lie algebra $\mathfrak{g}$ the *coinduced representation* of $\mathfrak{g}$ by ρ is the left $\mathscr{U}(\mathfrak{g})$-module $\operatorname{Hom}_{\mathscr{U}(\mathfrak{h})}(\mathscr{U}(\mathfrak{g}), W)$ where $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and W is an $\mathfrak{h}$-module with $\rho: \mathfrak{h} \to \operatorname{End}(W)$.

Let θ be the coinduced representation of $\mathfrak{g}$ by λ. Let Θ be the representation of $\mathscr{U}(\mathfrak{g})$ deduced from θ.

LEMMA 5.3. *Set* $V = \mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *and* $F = \mathrm{Coind}_{\mathfrak{p}}^{\mathfrak{g}}(0)$. *Consider* V *as a free* F*-module of rank* 1. *Then* $\Theta(\mathscr{U}(\mathfrak{g})) \subset \mathrm{Diff}(V)$.

Proof. In fact, we show that an element of $\mathfrak{g}$ can be seen as a differential operator of order 1 on V.

Let π be the coinduced representation $\pi: \mathfrak{g} \to \mathrm{End}(F)$. We know that $\mathfrak{g}$ acts on F by derivations (see Dixmier [D]). That means that $\pi(u)(fg) = (\pi(u)(f))g + f(\pi(u)(g))$ for $u \in \mathfrak{g}$ and $f, g \in F$.

For every $g \in F$ and $u \in \mathfrak{g}$, we have $(\mathrm{ad}\, g)(\pi(u)) = g\pi(u) - \pi(u)g$. For every $f \in F, (\pi(u)g)(f) = \pi(u)(gf) = (\pi(u)(g))f + g(\pi(u)(f))$. That implies $(\mathrm{ad}\, g)(\pi(u)) = \pi(u)(g) \in F$.

The elements of F are differential operators of order 0 since $(\mathrm{ad}\, g)(f) = gf - fg = 0$, $\forall f, g \in F$. Then it is clear that $(\mathrm{ad}\, f)(\mathrm{ad}\, g)(\pi(u)) = 0$, $\forall f, g \in F$. Henceforth, $\pi(u)$ is a differential operator of order 1 on F.

Since V is a free F-module of rank 1, it follows that $\theta(\mathfrak{g})$ are differential operators of order 1 on V also. That immediately implies that $\Theta(\mathscr{U}(\mathfrak{g}))$ is contained in $\mathrm{Diff}(V)$. Since $I(\lambda) = \mathrm{Ann}_{\mathscr{U}(\mathfrak{g})} L$ and $L^* \simeq V$, then we obtain $\Theta(\mathscr{U}(\mathfrak{g})/I(\lambda)) \subset \mathrm{Diff}(V)$. ∎

Remark 5.4. Observe that this lemma gives us an embedding of $\mathscr{U}(\mathfrak{g})/I(\lambda)$ in $\mathrm{Diff}(V)$. This immediately implies that $I(\lambda)$ is a completely prime ideal.

Conjecture 2. The image of the map **I** is exactly $\mathrm{Prim}_{GK}\mathscr{U}(\mathfrak{g})$ where $\mathrm{Prim}_{GK}\mathscr{U}(\mathfrak{g})$ denotes the set of primitive ideals I of $\mathscr{U}(\mathfrak{g})$ such that $\mathscr{U}(\mathfrak{g})/I$ has finite Gelfand–Kirillov dimension.

REFERENCES

[AM] M. F. Atiyah and I. G. Macdonald, "Introduction to Commutative Algebra," Addison–Wesley, Reading, MA, 1969.

[Be] K. Bennani, "Sur certaines représentations irréductibles d'algèbres de Lie de champs de vecteurs," Thèse de Doctorat, Université Paris VII, Fevrier 1993.

[BGR] W. Borho, P. Gabriel, and R. Rentschler, Primideale in Einhüllenden ausflösbarer Lie-Algebren, *in* Lecture Notes in Mathematics, Vol. 357, Springer-Verlag, New York/Berlin, 1973.

[BK] W. Borho and H. Kraft, Über die Gelfand–Kirillov Dimension, *Math. Ann.* **220** (1976), 1–24.

[Bo] N. Bourbaki, "Eléments de Mathématiques, Algèbre commutative," Chaps. 1 and 2, Fascicule XXVII, Hermann, Paris, 1961.

[C] J. Y. Charbonnel, Sur la méthode des orbites pour une algèbre de Lie résoluble, preprint.

[CD] N. Conze and M. Duflo, Sur l'algèbre enveloppante d'une algèbre de Lie résoluble, *Bull. Sci. Math.* **94** (1970), 201–208.

[D] J. Dixmier, "Algèbres enveloppantes," Cahiers Scientifiques, Fascicule XXXVII, Gauthier–Villars, Paris, 1974.

[Du] M. Duflo, Sur les représentations irréductibles des algèbres de Lie contenant un idéal nilpotent, *C. R. Acad. Sci. Paris Sér. A* **270** (1970), 504–506.

[GK] I. Gelfand and A. Kirillov, Sur les corps liés aux algèbres enveloppantes des algèbres de Lie, *Inst. Hautes Études Sci. Publ. Math.* **31** (1966), 5–19.

[H] R. Hartshorne, "Algebraic Geometry," Springer-Verlag, New York/Berlin, 1977.

[J] D. A. Jordan, On the ideals of a Lie algebra of derivations, *J. London Math. Soc.* (2) **33** (1986), 33–39.

[KL] G. R. Krause and T. H. Lenagan, Growth of algebras and Gelfand–Kirillov dimension, *in* Research Notes in Mathematics, Vol. 116, Pitman, London, 1985.

[L] S. Lang, "Algebra," 3rd ed., Addison–Wesley, Reading, MA, 1993.

[M] O. Mathieu, Bicontinuity of the Dixmier map, *J. Amer. Math. Soc.* **4**, No. 4 (1991), 837–863.

[Ma] H. Matsumura, "Commutative Ring Theory," Cambridge Univ. Press, Cambridge, UK, 1986.

[R] R. Rentschler, L'injectivité de l'application de Dixmier pour les algèbres de Lie résolubles, *Invent. Math.* **23** (1974), 49–71.

[S] I. R. Shafarevich, "Basic Algebraic Geometry, I," Springer-Verlag, New York/Berlin, 1994.