Letters in Mathematical Physics **66:** 141–155, 2003.

Back to the Amitsur–Levitzki Theorem: a Super Version for the Orthosymplectic Lie Superalgebra $\mathfrak{osp}(1, 2n)$

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(Received: 25 September 2003; final version: 20 October 2003)

Abstract. We prove an Amitsur–Levitzki type theorem for the Lie superalgebras $\mathfrak{osp}(1, 2n)$ inspired by Kostant's cohomological interpretation of the classical theorem. We show that the Lie superalgebras $\mathfrak{gl}(p, q)$ cannot satisfy an Amitsur–Levitzki type super identity if $pq \neq 0$ and conjecture that neither can any other classical simple Lie superalgebra with the exception of $\mathfrak{osp}(1, 2n)$.

Mathematics Subject Classifications (2000). 17B20, 17B56.

Key words. Amitsur–Levitzki theorem, Lie superalgebras, transgression operator.

0. Introduction

The Amitsur–Levitzki theorem states that $\mathfrak{gl}(n)$ satisfies the standard polynomial identity of order $2n$. More precisely, we have the following theorem:

THEOREM. *Define for $X_i \in \mathfrak{gl}(n), k \geqslant 1$:*

$$I_k(X_1, \ldots, X_k) := \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) X_{\sigma(1)} \ldots X_{\sigma(k)}.$$

Then $I_{2n} = 0$. (It is easy to see that $I_k \neq 0$ if $k < 2n$ and from $I_{2n} = 0$, that $I_k = 0$ if $k > 2n$ [7]).

Amitsur and Levitzki proved their theorem using an inductive method that does not explain why such identity exists [1, 7]. Later, several simplifications and improvements of their proof, including graphical ones and several new proofs were given [7, 10, 12, 19–21, 23]. However, all of these proofs except Kostant's lack a real interpretation of the result.

Eight years after Amitsur and Levitzki, Kostant published a truly beautiful proof of their theorem, based on the cohomology of Lie algebras [12]. Besides explaining the existence of the theorem, he proved with his method that $\mathfrak{o}(2n)$ satisfies the standard polynomial identity of order $4n-2$ (as a consequence of the particular structure of its invariants due to the existence of the Pffafian). Another proof of this result was later obtained by Rowen using a direct method, but with some difficulties [21]. Finally in 1981, Kostant [10] closed the subject once and for all by providing a very nice interpretation of the theorem in the context of representation theory and generalizing it using his separation of variables theorem [11]. To our knowledge, no one has returned to the Amitsur–Levitzki theorem since then.

A few comments can be made about Kostant's proofs of the Amitsur–Levitzki theorem. First, both proofs use the polynomial structure of the ring of invariants of a semi simple Lie algebra. Second, his cohomological proof is based on a quite sophisticated theorem of cohomology of Lie algebras (namely, the Hopf–Koszul–Samelson theorem, see e.g. [9]), of which the Amitsur–Levitzki theorem is a consequence, modulo some combinatorial identities concerning the trace [12]. We can give a more economical proof based on similar arguments, but that does not rely on the Hopf–Koszul–Samelson theorem. Our proof uses only elementary properties of the Chevalley–Cartan transgression operator [3, 4, 9] and some identities concerning the invariants $\mathrm{Tr}(X^k)$. It will not be presented in this Letter. However, a completely similar reasoning will allow us to handle the orthosymplectic case $\mathfrak{osp}(1,2n)$.

The goal of this Letter is to study possible versions of the Amitsur–Levitzki theorem in the case of Lie superalgebras. Consider the Lie super algebra $\mathfrak{gl}(p,q)$ and define for $X_1,\ldots,X_k\in\mathfrak{gl}(p,q)$:

$$\mathscr{A}_k(X_1,\ldots,X_k):=\sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})X_{\sigma(1)}\ldots X_{\sigma(k)},$$

where the super sign $\varepsilon(\sigma,\mathscr{X})$ will be defined in Section 1. The polynomial $\mathscr{A}_k$ is invariant under the action of the super algebra $\mathfrak{gl}(p,q)$. We call $\mathscr{A}_k$ the standard super polynomial of order k and it is clear that this polynomial is a natural candidate to replace I_k in the case of the superalgebra $\mathfrak{gl}(p,q)$. The next step is to check whether $\mathscr{A}_k$ is zero for k sufficiently big. However, if $pq\neq 0$, one can easily see that this is not true: there always exists a non-nilpotent element $X\in\mathfrak{gl}(p,q)_{\bar{1}}$ and, since $\mathscr{A}_k(X,\ldots,X)=k!X^k$, it results that $\mathscr{A}_k\neq 0$ for all k. Therefore there is no standard super identity for $\mathfrak{gl}(p,q)$. With this counter-example in mind, one might think there is little hope in finding such an identity for the simple subalgebras of $\mathfrak{gl}(p,q)$.

Nevertheless, a closer look at the counter-example shows that it should be translated in terms of invariants: the algebra of invariants of $\mathfrak{gl}(p,q)$ is not finitely generated [8, 22]. If we follow the philosophy of Kostant's proofs, the algebra of invariants of the considered Lie superalgebra should be a polynomial algebra, which leaves us with a single choice: $\mathfrak{osp}(1,2n)$. For this series of Lie superalgebras, the algebra of invariants is a polynomial algebra in n variables (see [16, 22]). In addition, it is easy to see that all elements in $\mathfrak{osp}(1,2n)_{\bar{1}}$ are nilpotent (see Section 2), so the counter-example above does not apply.

As a consequence, the series $\mathfrak{osp}(1, 2n)$ seems to be a good candidate for an Amitsur–Levitzki super theorem and our goal in this Letter is to show that this super version does exists. The main result presented here is the following:

THEOREM. *For* $X_1, \ldots, X_{4n+2} \in \mathfrak{osp}(1, 2n)$, $\mathscr{A}_{4n+2}(X_1, \ldots, X_{4n+2}) = 0$.

Notice that the number $4n + 2$ appearing in the above theorem is precisely the one for $\mathfrak{gl}(2n + 1)$ in the classical case of the Amitsur–Levitzki theorem.

As we mentioned before, the proof of this theorem follows the lines of Kostant's cohomological proof [12], but in a simpler form. Our proof does not need to use a powerful theorem such as that of Hopf, Koszul and Samelson for $\mathfrak{osp}(1, 2n)$ (see [6]), but only elementary properties of a (super) transgression operator and some identities concerning super traces.

We believe that in general there is no super Amitsur–Levitzki theorem for the classical Lie superalgebras, with the exception of the series $\mathfrak{osp}(1, 2n)$. This can be explained by the fact that their algebra of invariants is not (in general) finitely generated. Recall that $\mathfrak{osp}(1, 2n)$ are the only simple Lie superalgebras (together with simple Lie algebras) that are also semisimple [5] (meaning complete reducibility of finite-dimensional representations). The fact that these Lie superalgebras satisfy an identity of the Amitsur–Levitzki type strengthens the impression that they are very close to simple Lie algebras. However, the existence of a ghost center and of exotic primitive ideals in the enveloping algebra [15–17] indicate that the analogy cannot be carried much further.

COMMENTS AND PERSPECTIVES

(1) We want to stress that the present study was performed within the context of the theory of invariants of Lie superalgebras and in that spirit. It would of course be interesting to relate our super identity with the general theory of PI-algebras (a very active domain, see, e.g., [2, 18, 21]) where the classical Amitsur–Levitzki theorem plays an important role. That is a different study which remains to be carried out since our super identity does not seem to appear in the PI-algebras literature.

(2) In Kostant [12], the standard polynomial for $\mathfrak{gl}(n)$ is translated in terms of the symmetric and alternating groups, and as a consequence it is proved that the Amitsur–Levitzki theorem implies a deep classical theorem of Frobenius. The super Amitsur–Levitzki identity does not hold for $\mathfrak{gl}(p, q)$. Nevertheless, it would be very interesting to investigate how the canonical pairing between $\mathrm{GL}(n)$ and the symmetric group can be extended to the super case and what, within that framework, would be the meaning of the standard super polynomial.

(3) We conjecture that a super Amitsur–Levitzki type identity holds not only for the canonical representation of $\mathfrak{osp}(1, 2n)$ (as proved in the present paper), but also for any finite-dimensional representation. In order to prove such a result, one should adapt the strategy developed in Kostant [10]. The Separation Theorem for $\mathfrak{osp}(1, 2n)$

is needed, and it holds thanks to Musson [16]. But one also needs an analogue of Kostant's characterization of harmonics [11]. To our knowledge, this has never been done for $\mathfrak{osp}(1, 2n)$ and is, in itself, a very interesting challenge.

1. Notations

1.1. ALGEBRAS OF SUPERSYMMETRIC AND SKEW SUPERSYMMETRIC MULTILINEAR MAPPINGS

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a finite-dimensional $\mathbb{Z}_2$-graded vector space. Considered elements $X \in V$ are supposed homogeneous, and we denote by a small x the degree. On $W = \mathbb{C}$, set $W_{\bar{0}} = \mathbb{C}$ and $W_{\bar{1}} = \{0\}$. Let $\mathscr{F}(V)$ be the $\mathbb{Z}$-graded space of multilinear forms on V and $\mathscr{F}^p(V)$ the subspace of p-forms. Consider the natural $\mathbb{Z}_2$-grading on $\mathscr{F}^p(V)$:

$$F \in \mathscr{F}^p(V), \qquad \deg_{\mathbb{Z}_2}(F) = f \quad \text{iff} \quad \deg_{\mathbb{Z}_2}(F(X_1, \ldots, X_p)) = x_1 + \cdots + x_p + f$$

The space $\mathscr{F}(V)$ is endowed with the usual tensor product $\otimes$, and with a super tensor product denoted by $\underset{s}{\otimes}$ and defined as

$$(F \underset{s}{\otimes} G)(X_1, \ldots, X_{p+q}) := (-1)^{g(x_1+\cdots+x_p)} F(X_1, \ldots, X_p) G(X_{p+1}, \ldots, X_{p+q}),$$

for $X_1, \ldots, X_{p+q} \in V$, $F \in \mathscr{F}^p_f(V)$, $G \in \mathscr{F}^q_g(V)$ with $\deg_{\mathbb{Z}_2}(F) = f$ and $\deg_{\mathbb{Z}_2}(G) = g$.

Let $\mathscr{X} = (X_1, \ldots, X_p) \in V^p$ and σ an element of the symmetric group $\mathfrak{S}_p$. Define $\varepsilon(\sigma, \mathscr{X}) := (-1)^{K(\sigma, \mathscr{X})}$, where $K(\sigma, \mathscr{X}) := \sharp\{(i,j) \mid X_{\sigma(i)}, X_{\sigma(j)} \in V_{\bar{1}}, i < j \text{ and } \sigma(i) > \sigma(j)\}$. It follows from the definition that $\varepsilon(\sigma, \mathscr{X})$ is a multiplier, that is

$$\varepsilon(\sigma\sigma', \mathscr{X}) = \varepsilon(\sigma, \mathscr{X})\varepsilon(\sigma', \sigma^{-1} \cdot \mathscr{X}),$$

with $\sigma \cdot \mathscr{X} := (X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(p)})$.

We can consider three actions of $\mathfrak{S}_p$ on $\mathscr{F}^p(V)$:

$$\begin{aligned}
&\sigma \cdot F(X_1, \ldots, X_p) := F(X_{\sigma(1)}, \ldots, X_{\sigma(p)}),\\
&\sigma_{\dot{s}} F(X_1, \ldots, X_p) := \varepsilon(\sigma, \mathscr{X}) F(X_{\sigma(1)}, \ldots, X_{\sigma(p)}),\\
&\sigma_{\dot{a}} F(X_1, \ldots, X_p) := \varepsilon(\sigma)\varepsilon(\sigma, \mathscr{X}) F(X_{\sigma(1)}, \ldots, X_{\sigma(p)}).
\end{aligned}$$

We then say that a p-form F is *supersymmetric* if $\sigma_{\dot{s}} F = F$, $\forall\, \sigma \in \mathfrak{S}_p$ and *skew supersymmetric* if $\sigma_{\dot{a}} F = F$, $\forall\, \sigma \in \mathfrak{S}_p$. We denote by $\mathscr{S}(V)$ the space of supersymmetric forms and by $\mathscr{A}(V)$ the space of skew supersymmetric forms.

Now let S and A be two operators on $\mathscr{F}(V)$ defined as:

$$\mathrm{S}(F) := \sum_{\sigma \in \mathfrak{S}_p} \sigma_{\dot{s}} F, \qquad \mathrm{A}(F) := \sum_{\sigma \in \mathfrak{S}_p} \sigma_{\dot{a}} F, \quad \forall\, F \in \mathscr{F}^p(V).$$

We can then define a product on $\mathscr{P}(V)$ and $\mathscr{A}(V)$ as

$$F \cdot G := \frac{1}{p!q!} \mathrm{S}(F \underset{s}{\otimes} G),$$

for $F \in \mathscr{P}^p(V)$, $G \in \mathscr{P}^q(V)$,

$$F \wedge G := \frac{1}{p!q!} \mathrm{A}(F \underset{s}{\otimes} G),$$

for $F \in \mathscr{A}^p(V)$, $G \in \mathscr{A}^q(V)$.

This gives an algebra structure on $\mathscr{P}(V)$ and $\mathscr{A}(V)$. The algebra $\mathscr{P}(V)$ is $\mathbb{Z}_2$-graded (since V is $\mathbb{Z}_2$-graded) and isomorphic to the (usual) tensor product $\mathrm{Sym}(V_{\bar{0}}^*) \otimes \mathrm{Ext}(V_{\bar{1}}^*)$. The algebra $\mathscr{A}(V)$ is double graded by $\mathbb{Z} \times \mathbb{Z}_2$, and isomorphic to $\mathrm{Ext}(V_{\bar{0}}^*) \otimes_{\mathbb{Z}\times\mathbb{Z}_2} \mathrm{Sym}(V_{\bar{1}}^*)$. We have

$$F \cdot G = (-1)^{fg} G \cdot F$$

for $F, G \in \mathscr{P}(V)$, $\deg_{\mathbb{Z}_2}(F) = f$, $\deg_{\mathbb{Z}_2}(G) = g$, and

$$F \wedge G = (-1)^{nm+fg} G \wedge F,$$

for $F, G \in \mathscr{A}(V)$, $\deg_{\mathbb{Z}\times\mathbb{Z}_2}(F) = (n, f)$, $\deg_{\mathbb{Z}\times\mathbb{Z}_2}(G) = (m, g)$.

These relations imply that $\mathscr{P}(V)$ and $\mathscr{A}(V)$ are supercommutative with respect to their gradation. We can say that $\mathscr{P}(V)$ (respectively, $\mathscr{A}(V)$) is the analogous of the algebra of polynomial functions (respectively, of the Grassman algebra) in the nongraded case.

The following formulae will be useful in this work: let $\phi_1, \ldots, \phi_p \in V^*$ with $\mathbb{Z}_2$-degrees $\varphi_1, \ldots, \varphi_p$, $\varphi := (\varphi_1, \ldots, \varphi_p)$, then

$$\phi_1 \cdot \ldots \cdot \phi_p = (-1)^{\Omega(\varphi,\varphi)} \mathrm{S}(\phi_1 \otimes \cdots \otimes \phi_p)$$

and

$$\phi_1 \wedge \cdots \wedge \phi_p = \mathrm{A}(\phi_1 \underset{s}{\otimes} \cdots \underset{s}{\otimes} \phi_p) = (-1)^{\Omega(\varphi,\varphi)} \mathrm{A}(\phi_1 \otimes \cdots \otimes \phi_p),$$

where Ω is the 2-form with the matrix

$$\begin{pmatrix} 0 & \ldots & \ldots & 0 \\ 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \ldots & 1 & 0 \end{pmatrix}.$$

For $X \in V$, define super derivations D_X and ι_X of $\mathscr{P}(V)$ and $\mathscr{A}(V)$ respectively as

$$\mathrm{D}_X(F)(X_1, \ldots, X_{p-1}) := (-1)^{xf} F(X, X_1, \ldots, X_{p-1})$$

for $F \in \mathscr{P}(V)$, $\deg_{\mathbf{Z}_2}(F) = f$ and

$$\imath_X(F)(X_1, \ldots, X_{p-1}) := (-1)^{xf} F(X, X_1, \ldots, X_{p-1})$$

for $F \in \mathscr{A}(V)$, $\deg_{\mathbf{Z}\times\mathbf{Z}_2}(F) = (p, f)$.

Hence, D_X is a super derivation of degree x of $\mathscr{P}(V)$ and $\imath_X$ is a super derivation of degree $(-1, x)$ of $\mathscr{A}(V)$.

1.2. COHOMOLOGY OF LIE SUPERALGEBRAS (see [14])

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra with $\dim \mathfrak{g}_{\bar{0}} = p$ and $\dim \mathfrak{g}_{\bar{1}} = q$. The contragredient representation $\check{\mathrm{a}}\mathrm{d}$ of the adjoint representation ad can be extended to a representation L^s of $\mathfrak{g}$ into $\mathscr{P}(\mathfrak{g})$ and to a representation L^a of $\mathfrak{g}$ into $\mathscr{A}(\mathfrak{g})$. For $X \in \mathfrak{g}$, L^s_X (resp. L^a_X) is the super derivation of degree x (resp. $(0, x)$) of $\mathscr{P}(\mathfrak{g})$ (resp. $\mathscr{A}(\mathfrak{g})$) defined as: for $F \in \mathscr{P}(\mathfrak{g})$ (resp. $\mathscr{A}(\mathfrak{g})$) with $\deg_{\mathbf{Z}_2}(F) = f$ (resp. $\deg_{\mathbf{Z}\times\mathbf{Z}_2}(F) = (n, f)$),

$$\mathrm{L}^{a,s}_X F(X_1, \ldots, X_n) := -(-1)^{xf} \sum_{j=1}^{n} (-1)^{x(x_1+\cdots+x_{j-1})} F(X_1, \ldots, \mathrm{ad}\, X(X_j), \ldots, X_n).$$

Denote by $I^s(\mathfrak{g})$ and $I^a(\mathfrak{g})$ the invariants under these actions. Let d be the map from $\mathfrak{g}^*$ to $\mathscr{A}(\mathfrak{g})$ defined as:

$$d\phi(X_1, X_2) := -\phi([X_1, X_2]), \quad \forall\ \phi \in \mathfrak{g}^*.$$

There exists a super derivation (also denoted by d) of $\mathscr{A}(\mathfrak{g})$ of degree $(1, 0)$ extending d: for $F \in \mathscr{A}(\mathfrak{g})$,

$$dF(X_1, \ldots, X_{n+1}) := \sum_{i<j} (-1)^{i+j} (-1)^{x_i(x_1+\cdots+x_{i-1})} (-1)^{x_j(x_1+\cdots+\hat{x}_i+\cdots+x_{j-1})} F([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{n+1}).$$

From the Jacobi identity, it comes $d^2 = 0$ and we can then define the cohomology (with trivial coefficients) of $\mathfrak{g}$ as

$$Z(\mathfrak{g}) := \mathrm{Ker}(d), \qquad B(\mathfrak{g}) := \mathrm{Im}(d) \quad \text{and} \quad H(\mathfrak{g}) := Z(\mathfrak{g})/B(\mathfrak{g}).$$

Let $\{X_1, \ldots, X_{p+q}\}$ be a basis of $\mathfrak{g}$ and $\{\phi_1, \ldots, \phi_{p+q}\}$ its dual basis. Define the forms $\tilde{\phi}_i$ as $\tilde{\phi}_i(X) := (-1)^{x_i x} \phi_i(X)$, $X \in \mathfrak{g}$. Thus, one has

$$d = \frac{1}{2} \sum_{i=1}^{p+q} \tilde{\phi}_i \wedge \mathrm{L}^a_{X_i}. \tag{1.1}$$

It results from (1.1) that $I^a(\mathfrak{g}) \subset \mathrm{Z}(\mathfrak{g})$. Moreover, one has

$$\mathrm{L}^a_X = \imath_X \circ d + d \circ \imath_X, \quad \forall\ X \in \mathfrak{g}. \tag{1.2}$$

As a consequence, L^a_X commutes with d and $\mathrm{L}^a_X(Z(\mathfrak{g})) \subset B(\mathfrak{g})$.

2. Orthosymplectic Lie Superalgebras

In this section, let $\mathfrak{g}$ be the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$. Among simple Lie superalgebras, the orthosymplectic $\mathfrak{osp}(1, 2n)$ are the only ones (together with simple Lie algebras) satisfying the remarkable property of being semisimple [5], meaning that every finite-dimensional representation is completely reducible.

2.1. THE WEYL ALGEBRA AND $\mathfrak{osp}(1, 2n)$

In the quantization framework, the Lie superalgebra $\mathfrak{g}$ can be realized as follows: let A_n be the Weyl algebra generated by $\{p_i, q_i, i = 1, \ldots, n\}$ with

$$[p_i, q_i]_{\mathscr{L}} = 1, \quad \forall\ i, \quad [p_i, q_j]_{\mathscr{L}} = [p_i, p_j]_{\mathscr{L}} = [q_i, q_j]_{\mathscr{L}} = 0,$$

if $i \neq j$, where $[\cdot, \cdot]_{\mathscr{L}}$ denotes the Lie bracket. The algebra A_n is $\mathbb{Z}_2$-graded, hence a Lie superalgebra. Denote by $[\cdot, \cdot]$ its bracket.

DEFINITION 2.1. The twisted adjoint action of A_n onto itself is defined as

$$\mathrm{ad}'A(B) := AB - (-1)^{a(b+1)}BA$$

for $A, B \in \mathsf{A}_n$, $\deg_{\mathbf{Z}_2}(A) = a$, $\deg_{\mathbf{Z}_2}(B) = b$.

Let $V_{\bar{1}} := \mathrm{span}\{p_i, q_i, i = 1, \ldots, n\}$ and $\mathfrak{h} := V_{\bar{1}} \oplus [V_{\bar{1}}, V_{\bar{1}}]$. Then $\mathfrak{h}$ is a subalgebra of the Lie superalgebra A_n. Let now $V := V_{\bar{0}} \oplus V_{\bar{1}}$ where $V_{\bar{0}} := \mathbb{C} \cdot 1$. We have $\mathrm{ad}'\,\mathfrak{h}(V) \subset V$. Moreover, the supersymmetric 2-form $F(X, Y) := [X, Y]_{\mathscr{L}}$, $X, Y \in V_{\bar{1}}$ and $F(1, 1) := -2$ is $\mathrm{ad}'\,\mathfrak{h}$-invariant. It follows that $\mathfrak{h} \simeq \mathfrak{osp}(1, 2n)$. An easy but remarkable consequence is the following:

PROPOSITION 2.2. *If $X \in \mathfrak{osp}(1, 2n)_{\bar{1}}$, then $X^3 = 0$.*

Proof. It is enough to show that if $X \in V_{\bar{1}}$, then $(\mathrm{ad}'X|_V)^3 = 0$. Using $(\mathrm{ad}'X)(1) = 2X$ and $(\mathrm{ad}'X)^2(Y) = 2[X, Y]_{\mathscr{L}}\, X, \forall\ Y \in V_{\bar{1}}$, the result follows. □

More generally:

PROPOSITION 2.3. *Let π be a finite-dimensional representation of $\mathfrak{g} = \mathfrak{osp}(1, 2n)$. If $X \in \mathfrak{g}_{\bar{1}}$, then $\pi(X)$ is nilpotent.*

Proof. We use here the realization of $\mathfrak{g}$ as $\mathfrak{h}$. Let $X \in \mathfrak{h}_{\bar{1}} = V_{\bar{1}}$, $X \neq 0$. There exists a Darboux basis of $V_{\bar{1}}$ for the form $F|_{V_{\bar{1}} \times V_{\bar{1}}}$ such that X is the first basis element. We can then suppose that $X = p_1$. Let $\mathfrak{l} = \mathfrak{l}_{\bar{0}} \oplus \mathfrak{l}_{\bar{1}}$ with $\mathfrak{l}_{\bar{1}} = \mathrm{span}\{p_1, q_1\}$ and $\mathfrak{l}_{\bar{0}} = [\mathfrak{l}_{\bar{1}}, \mathfrak{l}_{\bar{1}}]$. So $\mathfrak{l} \simeq \mathfrak{osp}(1, 2)$. Let $\rho = \pi|_{\mathfrak{l}}$. Write $\rho = \bigoplus_{i \in I} \rho_i$ its decomposition into simple components. If $d_0 = \max\{\dim \rho_i, i \in I\}$, then $\pi(p_1)^{d_0} = 0$. □

2.2. COHOMOLOGY OF $\mathfrak{osp}(1, 2n)$

From [5], the representation L^a of $\mathfrak{g}$ is completely reducible. Using Koszul's strategy in [13], this fact together with the results in Section 1.2, in particular, Equations (1.1) and (1.2), allows us to prove

LEMMA 2.4. *Every cohomology class of $H(\mathfrak{g})$ contains one and only one invariant cocycle. In particular, 0 is the unique invariant coboundary and $H(\mathfrak{g}) = I^a(\mathfrak{g})$.*

For the sake of completeness, we should mention that better results exist concerning $H(\mathfrak{g})$: Fuks and Leites [6] have announced that $H(\mathfrak{g}) \simeq H(\mathfrak{g}_{\bar{0}}) = H(\mathfrak{sp}(2n))$. However, we shall not need these results here.

2.3. INVARIANTS

Concerning $I^s(\mathfrak{g})$, it results from Kac's work that the Chevalley restriction theorem holds [16, 22]: let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and W the Weyl group, then the restriction of $I^s(\mathfrak{g})$ into $\mathrm{Sym}(\mathfrak{h}^*)^W$ is an algebra isomorphism. As a consequence, $I^s(\mathfrak{g})$ is a polynomial algebra in n variables. We will see later how to choose convenient generators.

3. Chevalley's Transgression Operator for Lie Superalgebras

The transgression operator t: $\mathrm{Sym}(\mathfrak{g}^*) \to \mathrm{Ext}(\mathfrak{g}^*)$ was introduced by Chevalley ([3, 4], see also [9]) and it is a fundamental tool in the theory of Lie algebras. In this section, we shall generalize this notion to the case of Lie superalgebras and give some elementary properties that will be useful in the sequel.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. Let $\{X_1, \ldots, X_p\}$ be a basis of $\mathfrak{g}_{\bar{0}}$, $\{Y_1, \ldots, Y_q\}$ a basis of $\mathfrak{g}_{\bar{1}}$, $\{\Omega_1, \ldots, \Omega_p\}$ and $\{\phi_1, \ldots, \phi_q\}$ their respective dual basis. There exists a super derivation R of $\mathscr{P}(\mathfrak{g})$ of degree 0 extending $\mathrm{Id}_{\mathfrak{g}^*}$:

$$\mathrm{R} := \sum_{i=1}^{p} \Omega_i \mathrm{D}_{X_i} - \sum_{j=1}^{q} \phi_j \mathrm{D}_{Y_j}.$$

We have:

$$\mathrm{R}(P) = (\deg_{\mathbb{Z}} P)\ P, \quad \forall P \in \mathscr{P}(\mathfrak{g}),$$

where $\deg_{\mathbb{Z}} P$ comes from $\mathscr{P}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}_{\bar{0}}^*) \otimes \mathrm{Ext}(\mathfrak{g}_{\bar{1}}^*)$ and from the natural $\mathbb{Z}$-gradations of $\mathrm{Sym}(\mathfrak{g}_{\bar{0}}^*)$ and $\mathrm{Ext}(\mathfrak{g}_{\bar{1}}^*)$.

There exists an algebra homomorphism $s: \mathscr{P}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ such that

$$s(\Omega_i) = d\Omega_i, \ \ i = 1, \ldots, p, \qquad \text{and} \qquad s(\phi_j) = d\phi_j, \ \ j = 1, \ldots, q$$

(since the $d\Omega_i$ $(i = 1, \ldots, p)$ commute, the $d\phi_j$ $(j = 1, \ldots, q)$ anticommute and the $d\Omega_i$, $d\phi_j$ $(i = 1, \ldots, p, j = 1, \ldots, q)$ commute).

One can easily check that $d(s(P)) = 0$, $\forall\ P \in \mathscr{P}(\mathfrak{g})$. Besides, s is a homomorphism of $\mathfrak{g}$-modules if $\mathscr{P}(\mathfrak{g})$ is endowed with the representation L^s and $\mathscr{A}(\mathfrak{g})$ endowed with the representation L^a. Therefore $s(I^s(\mathfrak{g})) \subset I^a(\mathfrak{g})$.

Following Chevalley, we now set:

DEFINITION 3.1. The transgression operator $t\colon \mathscr{P}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ is defined as

$$t(P) := \sum_{i=1}^{p} \Omega_i \wedge s(\mathrm{D}_{X_i}(P)) - \sum_{j=1}^{q} \phi_j \wedge s(\mathrm{D}_{Y_j}(P)), \quad \forall\ P \in \mathscr{P}(\mathfrak{g}) \tag{3.1}$$

A priori, this definition seems to be basis dependent, but this is not the case as we shall show below. For the time, let us state:

LEMMA 3.2. *One has* $d(t(P)) = s(\mathrm{R}(P))$, $\forall\ P \in \mathscr{P}(\mathfrak{g})$.

Since $\mathrm{R}(P) = (\deg_{\mathbb{Z}} P)\ P$, Lemma 3.2 shows that if P has no constant term, then $s(P)$ is a coboundary.

Moreover t is an s-derivation:

LEMMA 3.3. *One has* $t(P \cdot Q) = t(P) \wedge s(Q) + s(P) \wedge t(Q)$, *for all* $P, Q \in \mathscr{P}(\mathfrak{g})$.

In order to establish some other properties of the transgression, we need now an intrinsic definition of t. First, observe that there is an isomorphism $\mathrm{End}(\mathfrak{g}) = \mathfrak{g}^* \underset{s}{\otimes} \mathfrak{g}$ given by:

$$(\Omega \underset{s}{\otimes} X)(Y) := (-1)^{xy}\Omega(Y)\ X, \qquad \forall\ \Omega \in \mathfrak{g}^*, \quad X, Y \in \mathfrak{g}.$$

Thanks to this identification, the representation $\pi := \check{\mathrm{ad}} \otimes \mathrm{ad}$ becomes $\mathrm{ad}(\mathrm{ad}\cdot)$ and

$$\mathrm{Id}_{\mathfrak{g}} = \sum_{i=1}^{p} \Omega_i \underset{s}{\otimes} X_i - \sum_{j=1}^{q} \phi_j \underset{s}{\otimes} Y_j$$

is π-invariant.

Now fix $P \in \mathscr{P}(\mathfrak{g})$ and set $\tau_p\colon \mathrm{End}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ as

$$\tau_P(\Omega \underset{s}{\otimes} X) := \Omega \wedge s(\mathrm{D}_X(P)). \tag{3.2}$$

It is immediate that $\tau_P(\mathrm{Id}_{\mathfrak{g}}) = t(P)$, so the definition of t in (3.1) is basis independent. In addition, using the representation π on $\mathrm{End}(\mathfrak{g})$ and L^a on $\mathscr{A}(\mathfrak{g})$, one has

LEMMA 3.4. *If* $P \in I^s(\mathfrak{g})$, *then* $\tau_P\colon \mathrm{End}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ *is a* $\mathfrak{g}$-*module homomorphism.*

As a direct consequence of (3.2) and Lemma 3.4, we obtain

$$t(I^s(\mathfrak{g})) \subset I^a(\mathfrak{g}). \tag{3.3}$$

Combining 3.3, 1.1 and Lemma 3.2, we have

LEMMA 3.5. *Let* $I_+^s(\mathfrak{g})$ *be the subspace of* $I^s(\mathfrak{g})$ *with no constant terms. Then for all* $P \in I_+^s(\mathfrak{g})$, $s(P) = 0$.

Finally applying Lemma 3.3, we conclude

LEMMA 3.6. *For all* $P \in \mathbb{C} \oplus (I_+^s(\mathfrak{g}))^2$, $t(\mathrm{P}) = 0$.

Remark 3.7. For similar results in the nongraded case, see [4] or [9]. □

4. Standard Superpolynomials and Superidentities in $\mathfrak{gl}(p, q)$

In this section, $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with dim $V_{\bar{0}} = p$, dim $V_{\bar{1}} = q$, and $\mathfrak{g}$ is the Lie superalgebra $\mathfrak{g} = \mathrm{End}(V) \simeq \mathfrak{gl}(p, q)$.

We identify End(V) and $V \otimes V^*$ by using

$$Z \otimes \Omega(T) := Z.\Omega(T), \quad \forall\ Z,\ T \in V,\ \Omega \in V^*$$

Then define the super trace on $\mathfrak{g}$ as

$$\mathrm{str}(Z \otimes \Omega) := (-1)^{\omega z}\Omega(Z), \quad \forall\ Z \in V,\ \Omega \in V^*.$$

Remark 4.1. With this definition, the 2-form $B(Z|T) := \mathrm{str}(ZT)$ is supersymmetric and nondegenerate on $\mathfrak{g}$. In the case $p = 1$ and $q = 2n$, $B|_{\mathfrak{osp}(1,2n)}$ is nondegenerate as well. □

DEFINITION 4.2. The standard supersymmetric super polynomials $\mathscr{P}_k$ (resp. skew supersymmetric $\mathscr{A}_k$) are given by

$$\mathscr{P}_k(X_1, \dots, X_k) := \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma; \mathscr{X}) X_{\sigma(1)} \dots X_{\sigma(k)},$$
$$\mathscr{A}_k(X_1, \dots, X_k) := \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma)\varepsilon(\sigma; \mathscr{X}) X_{\sigma(1)} \dots X_{\sigma(k)},$$

where $k \geqslant 1$, $X_1, \dots, X_k \in \mathfrak{g}$.

The polynomials $\mathscr{P}_k$ and $\mathscr{A}_k$ are $\mathfrak{g}$-invariant k-linear maps from $\mathfrak{g}^k$ to $\mathfrak{g}$. They verify the recursive relations below:

$$\mathscr{P}_{k+1}(X_1, \dots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{x_j(x_1 + \cdots + x_{j-1})} X_j\ \mathscr{P}_k(X_1, \dots, \widehat{X_j}, \dots, X_{k+1}), \tag{4.2a}$$

$$\mathscr{A}_{k+1}(X_1, \dots, X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} (-1)^{x_j(x_1 + \cdots + x_{j-1})} X_j\ \mathscr{A}_k(X_1, \dots, \widehat{X_j}, \dots, X_{k+1}). \tag{4.2b}$$

From $\mathscr{P}_k$ and $\mathscr{A}_k$, we can construct $P_k \in I^s(\mathfrak{g})$ and $\Lambda_k \in I^a(\mathfrak{g})$:

$$\begin{aligned} P_k(X_1, \dots, X_k) &:= \operatorname{str}(\mathscr{P}_k(X_1, \dots, X_k)), \\ \Lambda_k(X_1, \dots, X_k) &:= \operatorname{str}(\mathscr{A}_k(X_1, \dots, X_k)). \end{aligned} \tag{4.3}$$

PROPOSITION 4.3. *One has:*

(a) $$\begin{aligned} &P_{2k+1}(X_1, \dots, X_{2k+1}) = (2k+1)B(\mathscr{P}_{2k}(X_1, \dots, X_{2k})|X_{2k+1}), \\ &\Lambda_{2k}(X_1, \dots, X_{2k}) = 0, \\ &\Lambda_{2k+1}(X_1, \dots, X_{2k+1}) = (2k+1)B(\mathscr{A}_{2k}(X_1, \dots, X_{2k})|X_{2k+1}). \end{aligned} \tag{4.4}$$

(b) $$\sum_{\sigma \in \mathfrak{S}_{2k}} \varepsilon(\sigma)\varepsilon(\sigma; \mathscr{X})[X_{\sigma(1)}, X_{\sigma(2)}] \dots [X_{\sigma(2k-1)}, X_{\sigma(2k)}] = 2^k \mathscr{A}_{2k}(X_1, \dots, X_{2k}) \tag{4.5}$$

(c) $$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_{2k+1}} \varepsilon(\sigma)\varepsilon(\sigma; \mathscr{X})[X_{\sigma(1)}, X_{\sigma(2)}] \dots [X_{\sigma(2j-1)}, X_{\sigma(2j)}] X_{\sigma(2j+1)} \times \\ &\qquad \times [X_{\sigma(2j+2)}, X_{\sigma(2j+3)}] \dots [X_{\sigma(2k)}, X_{\sigma(2k+1)}] = 2^k \mathscr{A}_{2k+1}(X_1, \dots, X_{2k+1}) \end{aligned} \tag{4.6}$$

Remark 4.4. The identities (4.4), (4.5) and (4.6) are super versions of classical identities in the non graded case. Their proofs are simple adaptations to the super case. Other super identities can be settled, but they will not be needed in this work. □

Let us examine what happens when we apply the transgression on the invariant P_k defined by the super trace.

THEOREM 4.5. *One has* $t(P_k) = (-1)^{k-1} k \Lambda_{2k-1}$.

Proof. The main argument here will be Lemma 3.3. Let M_{ij} be the coordinate forms. Then

$$M_{ii}(X_1 \dots X_k) = \sum_{R=(r_1, \dots, r_{k-1})} (-1)^{\Omega(m_{iR}, m_{iR})} M_{ir_1} \underset{s}{\otimes} \cdots \underset{s}{\otimes} M_{r_{k-1}i}(X_1, \dots, X_k)$$

where $m_{iR} := \begin{pmatrix} m_{ir_1} \\ \vdots \\ m_{r_{k-1}i} \end{pmatrix}$.

Supersymmetrizing, we obtain:

$$\begin{aligned} P_k = &\sum_{\substack{i \in [\![1,p]\!] \\ R}} (-1)^{\Omega(m_{iR}, m_{iR})} M_{ir_1} \cdot M_{r_1 r_2} \cdot \ldots \cdot M_{r_{k-1}i} - \\ &- \sum_{\substack{j \in [\![p+1,p+q]\!] \\ R}} (-1)^{\Omega(m_{jR}, m_{jR})} M_{jr_1} \cdot M_{r_1 r_2} \cdot \ldots \cdot M_{r_{k-1}j} \end{aligned}$$

(notice that the products above are calculated in $\mathscr{P}(\mathfrak{g})$).

From $t(M_{rs}) = M_{rs}$, $\forall\, r, s$ and Lemma 3.3, comes

$$t(M_{ir_1} \cdot \ldots \cdot M_{r_{k-1}i}) = \sum_{\ell=1}^{k} dM_{ir_1} \wedge dM_{r_1r_2} \wedge \cdots \wedge M_{r_{\ell-1}r_\ell} \wedge \cdots \wedge dM_{r_{k-1}i}$$

(if $\ell = k$ then $r_k = i$ in the sum).

Therefore:

$$\begin{aligned} &t(M_{ir_1} \cdot \cdots \cdot M_{r_{k-1}i})(X_1, \ldots, X_{2k-1}) \\ &\quad = (-1)^{\Omega(m_{iR}, m_{iR})} \frac{(-1)^{k-1}}{2^{k-1}} \sum_{\sigma,\ell} \varepsilon(\sigma)\varepsilon(\sigma, \mathscr{X}) M_{ir_1}([X_{\sigma(1)}, X_{\sigma(2)}]) \ldots \\ &\qquad \ldots M_{r_{\ell-1}r_\ell}(X_{\sigma(2\ell-1)}) \ldots M_{r_{k-1}i}([X_{\sigma(2k-2)}, X_{\sigma(2k-1)}]). \end{aligned}$$

Finally, we have

$$\begin{aligned} &\sum_R (-1)^{\Omega(m_{iR}, m_{iR})} t(M_{ir_1} \cdot \ldots \cdot M_{r_{k-1}i})(X_1, \ldots, X_{2k-1}) \\ &\quad = \frac{(-1)^{k-1}}{2^{k-1}} \sum_{\sigma,R,\ell} \varepsilon(\sigma)\varepsilon(\sigma, \mathscr{X}) M_{ir_1}([X_{\sigma(1)}, X_{\sigma(2)}]) \ldots \\ &\qquad\qquad \ldots M_{r_{\ell-1}r_\ell}(X_{\sigma(2\ell-1)}) \ldots M_{r_{k-1}i}([X_{\sigma(2k-2)}, X_{\sigma(2k-1)}]) \\ &\quad = (-1)^{k-1} \sum_\ell M_{ii}(\mathscr{A}_{2k-1}(X_1, \ldots, X_{2k-1})) \qquad \text{(by 4.6)} \\ &\quad = (-1)^{k-1} k M_{ii}(\mathscr{A}_{2k-1}(X_1, \ldots, X_{2k-1})). \end{aligned}$$

□

5. The Amitsur–Levitzki Theorem for $\mathfrak{osp}(1, 2n)$

Henceforth we will assume that $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ and $\tilde{\mathfrak{g}} = \mathfrak{gl}(1, 2n)$. We will now prove a (super) version of the Amitsur–Levitzki theorem for $\mathfrak{g}$. In other words, we will show:

THEOREM 5.1. *For all* $X_1, \ldots, X_{4n+2} \in \mathfrak{g}$, *we have* $\mathscr{A}_{4n+2}(X_1, \ldots, X_{4n+2}) = 0$.

Notice that this identity is valid if $X_1, \ldots, X_{4n+2} \in \mathfrak{g}_{\bar{0}}$ by the classical Amitsur–Levitzki theorem. Furthermore, if $X_1 = \cdots = X_{4n+2} = X \in \mathfrak{g}_{\bar{1}}$, then by Proposition 2.2, the identity holds as well.

The theorem will be a consequence of Theorem 4.5 and two lemmas:

LEMMA 5.2. *We have:*

(1) *For all* $X_1, \ldots, X_{2p+1} \in \mathfrak{g}$, $\mathscr{P}_{2p+1}(X_1, \ldots, X_{2p+1}) \in \mathfrak{g}$.
(2) *For all* $X_1, \ldots, X_{4p+1} \in \mathfrak{g}$, $\mathscr{A}_{4p+1}(X_1, \ldots, X_{4p+1}) \in \mathfrak{g}$.
(3) *For all* $X_1, \ldots, X_{4p+2} \in \mathfrak{g}$, $\mathscr{A}_{4p+2}(X_1, \ldots, X_{4p+2}) \in \mathfrak{g}$.

As a consequence, P_{2p+1}, Λ_{4p+1} and Λ_{4p+2} vanish as multilinear mappings on $\mathfrak{g}$.

Recall from Subsection 2.3 that the restriction $R\colon I^s(\mathfrak{g}) \to J$ is an algebra isomorphism where $J := \mathrm{Sym}(\mathfrak{h}^*)^W$. The elements of $\mathfrak{h}$ are the matrices

$$H(\alpha_1, \alpha_2, \ldots, \alpha_n) = \begin{pmatrix} \alpha_1 & 0 & & & \\ 0 & -\alpha_1 & & 0 & \\ & & \ddots & & \\ & 0 & & \alpha_n & 0 \\ & & & 0 & -\alpha_n \end{pmatrix}.$$

So one has $\mathrm{Sym}(\mathfrak{h}^*) = \mathbb{C}[\alpha_1, \ldots, \alpha_n]$ and the Weyl group is generated by permutations and changes signs of $\alpha_1, \ldots, \alpha_n$. For these reasons, we can write $J = \mathbb{C}[t_1, \ldots, t_n]$ where $t_k := \sum_{i=1}^n \alpha_i^{2k}$. It is clear that $t_k \in J$, $\forall\, k$ and that $t_k \in J_+^2$ if $k \geqslant n+1$ where J_+ denotes the augmentation ideal. On the other hand, $R(P_{2k}) = 2t_k$, therefore one deduces:

LEMMA 5.3. *One has* $I^s(\mathfrak{g}) = \mathbb{C}[P_2, P_4, \ldots, P_{2n}]$ *and* $P_{2n+2} \in (I_+^s(\mathfrak{g}))^2$.

We will next terminate the proof of Theorem 5.1.

Proof of Theorem 5.1. Let $t_{\mathfrak{g}}$ be the transgression defined on $\mathfrak{g}$ and $t_{\tilde{\mathfrak{g}}}$ be transgression defined on $\tilde{\mathfrak{g}}$. Since $\mathfrak{g}$ is a subalgebra of $\tilde{\mathfrak{g}}$, if P is a p-form in $\mathscr{P}(\tilde{\mathfrak{g}})$, one has $t_{\tilde{\mathfrak{g}}}(P)|_{\mathfrak{g}^p} = t_{\mathfrak{g}}(P|_{\mathfrak{g}^p})$. In the sequel, we use t for both transgressions $t_{\mathfrak{g}}$ and $t_{\tilde{\mathfrak{g}}}$, and we consider multilinear mappings restricted to $\mathfrak{g}$. Now, since $P_{2n+2} \in (I_+^s(\mathfrak{g}))^2$, we have $t(P_{2n+2}) = 0$ from Lemma 3.6. Using Theorem 4.5, we deduce $t(P_{2n+2}) = -(2n+2)\Lambda_{4n+3}$, hence $\Lambda_{4n+3} = 0$. From Proposition 4.3, for all $X_1, \ldots, X_{4n+3} \in \mathfrak{g}$,

$$\Lambda_{4n+3}(X_1, \ldots, X_{4n+3}) = (4n+3)B(\mathscr{A}_{4n+2}(X_1, \ldots, X_{4n+2})|X_{4n+3}).$$

But $\mathscr{A}_{4n+2}(X_1, \ldots, X_{4n+2}) \in \mathfrak{g}$ by Lemma 5.2(3), hence from Remark 4.1:

$$\mathscr{A}_{4n+2}(X_1, \ldots, X_{4n+2}) = 0, \text{for all } X_1, \ldots, X_{4n+2} \in \mathfrak{g}. \qquad \square$$

Remark 5.4. From (4.2b), we have $\mathscr{A}_k|_{\mathfrak{g}^k} = 0$ if $k \geqslant 4n+2$. Also one can check that $\mathscr{A}_{4n}|_{\mathfrak{g}_{\bar{0}}^{4n-1} \times \mathfrak{g}_{\bar{1}}} \neq 0$ (thanks to the Hopf–Koszul–Samelson theorem for $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2n)$). So the index obtained in Theorem 4.5 is the best possible, if one considers only even indices, a technical but justified assumption (see [10]). As for $\mathscr{A}_{4n+1}|_{\mathfrak{g}^{4n+1}}$, it does not vanish if $n = 1$ and $n = 2$, but the general case is still to be done.

Acknowledgements

We wish to thank D. Sternheimer and the referees for suggestion of several improvements to this Letter.

This Letter is dedicated to Moshé Flato (1937–1998). He introduced us to the Amitsur–Levitzki theorem (among many other subjects). When Moshé was a

first-year student at the Hebrew University in Jerusalem, Jakob Levitzki taught him algebra and (after his sudden death in 1956), was replaced in the middle of the year by his assistant Shimshon Avraham Amitsur, with whom Moshé considered working for a PhD before choosing physics with Giulio Racah. Moshé was an extraordinary mathematician and physicist, with insight in both fields. But above all, he was a superb humanbeing and a great friend.

References

1. Amitsur, A. S. and Levitzki, J.: Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449–463.
2. Benanti, F., Demmel, J., Drensky, V. and Koev, P.: Computational approach to polynomial identities of matrices, a survey, In: A. Giambruno (ed.), *Polynomial Identities and Combinatorial Methods*, Dekker, 2003.
3. Cartan, H.: La transgression dans un groupe de Lie et dans un espace fibré principal, *Coll. Topologie, CBRM Bruxelles*, (1950), 57–71.
4. Chevalley, C.: The Betti numbers of the exceptional Lie groups, *Proc. Intern. Congr. Math.* **II** (1950), 21–24.
5. Djokovic, D. Z. and Hochschild, G.: Semi-simplicity of 2-graded Lie algebras, *Illinois J. Math.* **20** (1976), 134–143.
6. Fuks, D. B. and Leites, D. H.: Cohomology of Lie superalgebras, *C. R. Acad. Bulg. Sci.* **37** (1984), 1595–1596.
7. Jacobson, N.: PI-Algebras, Lecture Notes in Math. 441, Springer-Verlag, Berlin, 1975.
8. Kantor, I. and Trishin, I.: The algebra of polynomial invariants of the adjoint representation of the Lie superalgebra $\mathfrak{gl}(m, n)$, *Comm. Algebra* **25**(7) (1997), 2039–2070.
9. Kostant, B.: Clifford analogue of the Hopf–Koszul–Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \mathrm{End}\, V_\rho \otimes C(P)$ and the $\mathfrak{g}$-module structure of $\bigwedge \mathfrak{g}$, *Adv. in Math.* **125** (1997), 275–350.
10. Kostant, B.: A Lie algebra generalization of the Amitsur–Levitzki theorem, *Adv. in Math.* **40** (1981), 155–175.
11. Kostant, B.: Lie groups representations on polynomial rings, *Amer. J. Math.* **23** (1963), 327– 404.
12. Kostant, B.: A theorem of Frobenius, a theorem of Amitsur–Levitzki and cohomology theory, *J. Math. Mech.* **7** (1958), 237–264.
13. Koszul, J. L.: Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. France* **78** (1950), 65–127.
14. Leites, D. H.: Cohomology of Lie superalgebras, *Funct. Anal. Appl.* **9** (1975), 75–76.
15. Musson, I.: Enveloping algebras of Lie super algebras, a survey, *Contemporary Math.* **124** (1992), 141–149.
16. Musson, I.: On the center of the enveloping algebra of a classical simple Lie superalgebras, *J. Algebra* **193** (1997), 75–101.
17. Pinczon, G.: The enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1, 2)$, *J. Algebra* **132** (1990), 219–242.
18. Razmyslov, Yu. P.: *Identities of Algebras and their Representations*, translated from the 1989 Russian original, Trans. Math. Monogr. 138, Amer. Math. Soc., Providence, RI, 1994.
19. Rosset, S.: A short proof of the Amitsur–Levitzki theorem, *Israel J. Math.* **23** (1976), 187–188.

20. Rowen, L. H.: Standard polynomials in matrix algebras, *Trans. Amer. Math.* **190** (1974), 253–284.
21. Rowen, L. H.: *Polynomials Identities in Ring Theory*, Academic Press, New York, 1980.
22. Sergeev, A. N.: The invariant polynomials on simple Lie superalgebras, *Represent. Theory* **3** (1999), 250–280.
23. Swan, R. G.: An application of graph theory to algebra, *Proc. Amer. Math. Soc.* **14** (1963), 367–373. Correction at *ibid.* **21** (1969), 397–380.