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Supertrace and Superquadratic Lie Structure on the Weyl Algebra, and Applications to Formal Inverse Weyl Transform

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Abstract. Using the Moyal $\star$-product and orthosymplectic supersymmetry, we construct a natural nontrivial supertrace and an associated nondegenerate invariant supersymmetric bilinear form for the Lie superalgebra structure of the Weyl algebra W. We decompose adjoint and twisted adjoint actions. We define a renormalized supertrace and a formal inverse Weyl transform in a deformation quantization framework and develop some examples.

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0. Introduction

Since Killing, it is known that the existence of a nondegenerate invariant symmetric bilinear form is a crucial property for a Lie algebra. Let us call such a Lie algebra a *quadratic* Lie algebra. These Lie algebras are of interest, not only when they are finite dimensional, but infinite dimensional as well [10]. A corresponding notion of *superquadratic* Lie superalgebra exists, namely, it is a Lie superalgebra endowed with a nondegenerate invariant supersymmetric bilinear form. Besides, invariant bilinear forms are often constructed from Traces or Supertraces (though it is not always the case), even in the infinite-dimensional case.

The main result of the present paper is:

THEOREM 1. *The Weyl algebra is a superquadratic Lie superalgebra with an essentially unique bilinear form derived from a supertrace.*

To explain the origin of this statement, let us consider the Weyl algebra W in $2n$ generators realized as the polynomial algebra in $2n$ indeterminates endowed with the Moyal $\star$-product denoted by $\star$. Briefly, in a deformation quantization framework (see [3,18,17]), we quantize the natural Poisson bracket in an explicit way (Moyal $\star$-product). The main advantage of this quantization is the fact of being invariant with respect to the natural embedding of the Lie superalgebra $\mathfrak{osp}(1,2n)$ in W [7,8,13], i.e. with respect to the orthosymplectic supersymmetry. The conjunction of both arguments, explicit Moyal product and supersymmetry, is indeed a powerful machinery that allows to deduce algebraic properties of the Weyl algebra, as we shall show in this paper.

The Weyl algebra W is a Lie algebra with bracket denoted by $[.,.]_{\mathscr{L}}$, and since it is naturally $\mathbb{Z}_2$-graded, W is a Lie superalgebra with bracket denoted by $[.,.]$.

We define the supertrace on the Weyl algebra W as the usual evaluation at 0:

$$\mathrm{Str}(F) := F(0), \quad \forall F \in \mathsf{W},$$

and the bilinear form κ on W by:

$$\kappa(F,G) := \mathrm{Str}(F \star G), \quad \forall F, G \in \mathsf{W}.$$

Then we prove:

THEOREM 2.

1. Str *is a supertrace on* W*, that is* $\mathrm{Str}([F,G]) = 0$*, for all* F*,* $G \in \mathsf{W}$ *and one has* $\ker(\mathrm{Str}) = [\mathsf{W},\mathsf{W}]$.
2. *The bilinear form* κ *is supersymmetric, nondegenerate and invariant under the adjoint representation of the Lie superalgebra* W.

Theorem 1 is a consequence of Theorem 2. Let us mention that obviously there is no nontrivial trace on the Lie algebra W.

As another consequence of Theorem 2, we deduce a decomposition of W:

$$\mathsf{W} = \mathbb{K} \oplus [\mathsf{W},\mathsf{W}],$$

recovering in a natural way a nice result of Musson [13].

Concerning the bilinear form κ, given the decomposition of W into homogeneous factors (for the commutative product), $\mathsf{W} = \bigoplus_{k \geqslant 0} \mathsf{S}^k$, we show that

$$\kappa(\mathsf{S}^k, \mathsf{S}^\ell) = \{0\}, \quad k \neq \ell,$$

so that the restriction of κ to each S^k is nondegenerate, and provides an explicit (orthogonal or symplectic, according to the parity of k) invariant bilinear form for the standard simple action of $\mathfrak{sp}(2n)$ on S^k.

We study in addition, the decomposition of the adjoint ad and twisted adjoint ad$'$ actions of the Lie superalgebra W on itself:

THEOREM 3.

1. *Under the adjoint action,* W *decomposes as* $\mathsf{W} = \mathbb{K} \oplus [\mathsf{W}, \mathsf{W}]$. *Moreover,* $[\mathsf{W}, \mathsf{W}]$ *is a simple* $\mathrm{ad}(\mathsf{W})$*-module.*
2. W *is a simple* $\mathrm{ad}'(\mathsf{W})$*-module and has a nondegenerate invariant supersymmetric bilinear form.*

The bilinear form associated to the ad'-action is deduced from κ, and extends the bilinear form used to construct the natural embedding of $\mathfrak{osp}(1, 2n)$ in W. As a corollary, we have:

COROLLARY. $[\mathsf{W}, \mathsf{W}]$ *is a simple superquadratic Lie superalgebra.*

Let us quote that the simplicity of $[\mathsf{W}, \mathsf{W}]$ has been known as a combination of a result of Montgomery [12] proving that $[\mathsf{W}, \mathsf{W}]/\mathbb{K} \cap [\mathsf{W}, \mathsf{W}]$ is simple and a result of Musson [13] proving that $\mathsf{W} = \mathbb{K} \oplus [\mathsf{W}, \mathsf{W}]$. Our proof, using a supertrace and the Moyal product, is direct and completely different from the initial proofs in [12] and [13].

Finally, we reinterpret the supertrace in a Renormalization Theory context. Briefly, let $\mathscr{P}$ be the polynomial algebra in n indeterminates: the Weyl algebra W acts on $\mathscr{P}$ as differential operators with polynomial coefficients. It has been shown in [14] that any linear operator on $\mathscr{P}$ is in fact a differential operator, possibly of infinite order. A slight improvement provides a very explicit remarkable formula:

THEOREM 4. *If* $T \in \mathscr{L}(\mathscr{P})$*, then*

$$T = \sum_N \frac{1}{N!} \Big(m \circ (T \otimes \mathscr{S}) \circ \Delta(x^N) \Big) \ \frac{\partial^N}{\partial x^N}$$

where m *is the product,* Δ *the coproduct and* $\mathscr{S}$ *the antipode of* $\mathscr{P}$.

Let Str be the ordinary supertrace defined on the ideal of finite-rank operators $\mathscr{L}_f(\mathscr{P})$ and $\mathrm{Str}_{\mathsf{W}}$ be the supertrace previously defined on W. We show the following theorem:

THEOREM 5. *Let* $T = \sum_I \alpha_I(Q) \star P^I$. *If* $T \in \mathscr{L}_f(\mathscr{P})$*, then:*

$$\mathrm{Str}(T) = \frac{1}{2^n} \sum_I \mathrm{Str}_{\mathsf{W}}(\alpha_I(Q) \star P^I).$$

Therefore we can construct a natural extension of Str to $\mathscr{L}_f(\mathscr{P}) \oplus \mathsf{W}$ (in fact, to a bigger subspace that will not be discussed here). For this renormalized extension that we denote RStr, one has:

$$\mathrm{RStr}(\mathrm{Id}) = \frac{1}{2^n},$$

a formula that renormalizes $\mathrm{Str}(\mathrm{Id}) = \infty - \infty$ obtained by brutally applying the definition of Str. Moreover, the power n in this formula is precisely the dimension of the underlying variety so our renormalization of the supertrace has a geometric flavor. At last, we give some ideas of what a formal inverse Weyl transform could be, and compute examples.

We have tried to make this paper as self-contained as possible. For instance, we give a short introduction to the Moyal $\star$-product in Section 1, and in Section 2, an introduction to the embedding of $\mathfrak{osp}(1, 2n)$ in W together with the decompositions of the corresponding adjoint and twisted adjoint actions. Not only because all this material provides tools used all along this paper, but also for the convenience of the reader, who is not expected to be an expert in deformation quantization theory, orthosymplectic supersymmetry, etc.

Remarks.

1. Theorem 2 results from Proposition 1.9 and Theorem 4.1. Theorem 3 corresponds to Theorem 3.7. Theorem 5.2 gives Theorem 4 and Theorem 5 is Theorem 5.4.
2. In Sections 1–4 of this paper, $\mathbb{K}$ denotes a field of characteristic zero (not necessarily algebraically closed). In Section 5, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
3. There are so many references on deformation quantization that we cannot quote all of them. The reader should refer to the beautiful papers [3] that are the beginning (and much more) of this theory and to [17, 18] for recent reviews with 164 references.

1. Moyal $\star$-Products

Moyal $\star$-products are the first examples of deformation quantization of Poisson brackets (see e.g. [3, 17, 18]). In this section, we recall their well-known properties, giving proofs for the convenience of the reader.

Let V be a vector space with a basis $\{P_1, Q_1, \ldots, P_n, Q_n\}$ and $\{p_1, q_1, \ldots, p_n, q_n\}$ its dual basis.

We denote by S the commutative algebra $\mathsf{S} := \mathrm{Sym}(V^*)$ with the usual grading $\mathsf{S} = \bigoplus_{k \geqslant 0} \mathsf{S}^k$ and the Poisson bracket:

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right), \quad \forall\, F, G \in \mathsf{S}.$$

Denoting by $\{X_1, \ldots, X_{2n}\}$ the given basis of V and by $\{x_1, \ldots, x_{2n}\}$ its dual basis, we introduce a duality that identifies S^* with the commutative algebra of formal power series $\mathscr{F} := \mathbb{K}[[X_1, \ldots, X_{2n}]]$ by:

$$\langle x^I \mid X^J \rangle := \delta_{I,J}\ I!$$

where $I,\ J \in \mathbb{N}^{2n}$, $x^I = x_1^{i_1} \ldots x_{2n}^{i_{2n}}$, $X^J = X_1^{j_1} \ldots X_{2n}^{j_{2n}}$ and $I! = i_1! \ldots i_{2n}!$.

The following properties result from a straightforward verification:

PROPOSITION 1.1.

1. *For all* $v \in V$ *and* $F \in \mathsf{S}$, $\langle F \mid e^v \rangle = F(v)$.
2. *For all* $v \in V$ *and* $F \in \mathsf{S}$,

$$\langle F \mid X^I e^v \rangle = \left\langle \frac{\partial^I F}{\partial x^I} \;\middle|\; e^v \right\rangle = \frac{\partial^I F}{\partial x^I}(v),$$

where $\dfrac{\partial^I F}{\partial x^I} := \dfrac{\partial^{i_1+\cdots+i_{2n}} F}{\partial x_1^{i_1} \cdots \partial x_{2n}^{i_{2n}}}$.

Notice that (1) is Taylor's formula and (2) means that the transpose of the operator $\frac{\partial^I}{\partial x^I}$ on S is the operator of multiplication by X^I on $\mathscr{F}$.

Define the operator $\wp\colon \mathsf{S}\otimes\mathsf{S} \to \mathsf{S}\otimes\mathsf{S}$ by:

$$\wp(F \otimes G) := \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \otimes \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \otimes \frac{\partial G}{\partial p_i} \right), \quad \forall F, G \in \mathsf{S}. \tag{1}$$

Since $\mathsf{S}\otimes\mathsf{S} = \mathbb{K}[p_1, q_1, \ldots, p_n, q_n, p'_1, q'_1, \ldots, p'_n, q'_n]$, we have:

$$(\mathsf{S}\otimes\mathsf{S})^* = \mathbb{K}[[P_1, Q_1, \ldots, P_n, Q_n, P'_1, Q'_1, \ldots, P'_n, Q'_n]].$$

If $D = {}^t\wp$, from Proposition 1.1 it follows:

$$D(A \otimes B) = \left(\sum_{i=1}^{n} (P_i \otimes Q_i - Q_i \otimes P_i) \right) \cdot\; A \otimes B, \quad \forall A, B \in \mathscr{F},$$

so that D is the operator of multiplication by $d(\mathscr{X}, \mathscr{X}') = \sum_{i=1}^{n} (P_i Q'_i - Q_i P'_i)$ on $(\mathsf{S}\otimes\mathsf{S})^*$ where $\mathscr{X} = (P_1, Q_1, \ldots, P_n, Q_n)$ and $\mathscr{X}' = (P'_1, Q'_1, \ldots, P'_n, Q'_n)$.

Denote by m the product of S. We can now define:

DEFINITION 1.2. For F, $G \in \mathsf{S}$, the *Moyal* $\star$*-product* is defined by:

$$F \underset{t}{\star} G := \left(m \circ \sum_{k \geqslant 0} \frac{t^k}{2^k\ k!}\ \wp^k \right) (F \otimes G).$$

Let Δ be the coproduct of $\mathscr{F}$ associated to m. With the usual obvious abuse of notation, one can write:

$$\Delta(A) = A(\mathscr{X} + \mathscr{X}'), \quad \forall A \in \mathscr{F}.$$

So if we denote by $m_{\underset{t}{\star}}$ the Moyal $\star$-product and by $\Delta_{\underset{t}{\star}}$ the associated coproduct $\Delta_{\underset{t}{\star}} = {}^t m_{\underset{t}{\star}}$, we get:

$$\Delta_{\underset{t}{\star}}(A) = \sum_{k \geqslant 0} \frac{t^k}{2^k\ k!}\ D^k(A(\mathscr{X} + \mathscr{X}')) = e^{\frac{t}{2} d(\mathscr{X}, \mathscr{X}')}\ A(\mathscr{X} + \mathscr{X}'), \quad \forall A \in \mathscr{F}.$$

PROPOSITION 1.3.

1. $\underset{t}{\Delta_\star}$ *is coassociative.*
2. $\underset{t}{m_\star}$ *is associative.*

Proof. Let $A \in \mathscr{F}$. One has:

$$\begin{aligned}(\mathrm{Id} \otimes \underset{t}{\Delta_\star})(\underset{t}{\Delta_\star}(A)) &= (\mathrm{Id} \otimes \underset{t}{\Delta_\star})\left(\mathrm{e}^{\frac{t}{2}d(\mathscr{X},\mathscr{X}')}\ A(\mathscr{X}+\mathscr{X}')\right)\\ &= \mathrm{e}^{\frac{t}{2}d(\mathscr{X}',\mathscr{X}'')}\,\mathrm{e}^{\frac{t}{2}d(\mathscr{X},\mathscr{X}'+\mathscr{X}'')}\ A(\mathscr{X}+\mathscr{X}'+\mathscr{X}'')\\ (\underset{t}{\Delta_\star} \otimes \mathrm{Id})(\underset{t}{\Delta_\star}(A)) &= \mathrm{e}^{\frac{t}{2}d(\mathscr{X},\mathscr{X}')}\ \underset{t}{\Delta_\star}(A)(\mathscr{X}+\mathscr{X}',\mathscr{X}'')\\ &= \mathrm{e}^{\frac{t}{2}d(\mathscr{X},\mathscr{X}')}\,\mathrm{e}^{\frac{t}{2}d(\mathscr{X}+\mathscr{X}',\mathscr{X}'')}\ A(\mathscr{X}+\mathscr{X}'+\mathscr{X}'')\end{aligned}$$

and since d is bilinear, one obtains $(\mathrm{Id}\otimes\underset{t}{\Delta_\star})\circ\underset{t}{\Delta_\star} = (\underset{t}{\Delta_\star}\otimes\mathrm{Id})\circ\underset{t}{\Delta_\star}$, so $\underset{t}{\Delta_\star}$ is coassociative. From $\underset{t}{\Delta_\star} = {}^t\underset{t}{m_\star}$, it follows that $\underset{t}{m_\star}$ is associative. □

We introduce the notation:

$$C_k(F,G) = \frac{1}{2^k k!}\left(m\circ\wp^k\right)(F\otimes G), \quad \forall\, F,G\in \mathsf{S}. \tag{2}$$

Then we can write:

$$F \underset{t}{\star} G = FG + \frac{t}{2}\{F,G\} + t^2 C_2(F,G) + \cdots, \quad \forall\, F,G\in\mathsf{S}.$$

Thus the Moyal $\star$-product is a deformation of the commutative product of S and also a deformation quantization of the Poisson bracket.

We remark that C_k is a bidifferential operator of order (k,k), so:

PROPOSITION 1.4. *For all* $F\in\mathsf{S}^f$ *and* $G\in\mathsf{S}^g$, $C_k(F,G)\in\mathsf{S}^{f+g-2k}$ (*with* $\mathsf{S}^\ell=\{0\}$ *when* $\ell<0$) *and*

$$F \underset{t}{\star} G = \sum_{k=0}^{\min(f,g)} t^k\ C_k(F,G).$$

Consider τ the twist $\tau(F\otimes G) := G\otimes F$, $\forall\, F,G\in\mathsf{S}$. Then $\wp\circ\tau = -\tau\circ\wp$, so $\wp^k\circ\tau = (-1)^k\tau\circ\wp^k$ and one deduces a useful parity property of the coefficients C_k:

$$C_k(G,F) = (-1)^k C_k(F,G), \quad \forall\, F,G\in\mathsf{S}. \tag{3}$$

As a consequence, the Lie bracket associated to the Moyal $\star$-product contains only odd terms:

$$[F,G]_{\underset{t}{\mathscr{L}}} = 2\sum_{p\geqslant 0} t^{2p+1}\ C_{2p+1}(F,G), \quad \forall\, F,G\in\mathsf{S}. \tag{4}$$

Now, from Proposition 1.4, we obtain the following useful property:

PROPOSITION 1.5. *Let* $F \in \mathsf{S}$ *of degree* $\leqslant 2$. *Then* $[F, G]_{\underset{t}{\mathscr{S}}} = t\{F, G\}$, *for all* $G \in \mathsf{S}$.

Therefore $[p_i, q_i]_{\underset{t}{\mathscr{S}}} = t$, for $i = 1, \ldots, n$ and all other brackets between p_i's and q_j's vanish.

Next we shall relate the Moyal $\star$-product to the Weyl algebra. For this purpose, we need a Lemma:

LEMMA 1.6. *One has*

1. *For all* $\varphi \in V^*$, $\varphi^{\underset{t}{\star} k} = \underbrace{\varphi \underset{t}{\star} \cdots \underset{t}{\star} \varphi}_{k \text{ times}} = \varphi^k$.
2. *For all* $\varphi_1 \cdots \varphi_k \in V^*$, $\varphi_1 \cdots \varphi_k = \dfrac{1}{k!} \displaystyle\sum_{\sigma \in \mathfrak{S}_k} \varphi_{\sigma(1)} \underset{t}{\star} \cdots \underset{t}{\star} \varphi_{\sigma(k)}$.

Proof. To prove (1) of the above lemma: by Proposition 1.4, one has $\varphi \underset{t}{\star} \varphi = \varphi^2 + \frac{t}{2}\{\varphi, \varphi\} = \varphi^2$. Also $\varphi \underset{t}{\star} \varphi \underset{t}{\star} \varphi = \varphi^3 + \frac{t}{2}\{\varphi, \varphi^2\} = \varphi^3$ and so on.

Then $(\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k)^{\underset{t}{\star} k} = (\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k)^k$, for all $\lambda_1, \ldots, \lambda_k \in \mathbb{K}$. Identifying the coefficient of the term $\lambda_1 \ldots \lambda_k$ on each side, one obtains:

$$\sum_{\sigma \in \mathfrak{S}_k} \varphi_{\sigma(1)} \underset{t}{\star} \cdots \underset{t}{\star} \varphi_{\sigma(k)} = \sum_{\sigma \in \mathfrak{S}_k} \varphi_{\sigma(1)} \cdots \varphi_{\sigma(k)}$$

and (2) follows. □

Remark 1.7. Consequently, since S^k is linearly generated by $\{\varphi^k \mid \varphi \in V^*\}$, we can conclude that $(\mathsf{S}, m_{\underset{t}{\star}})$ is generated by V^* as an algebra.

Let us denote by $\mathsf{W}(n)$, or W when there is no ambiguity, the Weyl algebra generated by $\{p_1, q_1, \ldots, p_n, q_n\}$ with relations $p_iq_j - q_jp_i = \delta_{ij}$, $p_ip_j - p_jp_i = q_iq_j - q_jq_i = 0$, for $i, j = 1, \ldots, n$ (see [6]). A well-known fact is that W is the algebra of polynomial differential operators on $\mathbb{K}[y_1, \ldots, y_n]$ with $p_i = \frac{\partial}{\partial y_i}$ and $q_i = y_i \times \cdot$. We can also realize W as the quotient algebra $T(V^*)/I$ where I is the ideal in the tensor algebra $T(V^*)$ generated by the above relations. Therefore we can consider that $V^* \subset \mathsf{W}$ as will be done in the sequel.

Denote by $\rho\colon \mathsf{S} \to \mathsf{W}$ the *symmetrization map* defined by:

$$\rho(\varphi_1 \cdots \varphi_k) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varphi_{\sigma(1)}\varphi_{\sigma(2)} \cdots \varphi_{\sigma(k)}$$

where in the right-hand side, the products are computed in W. Notice that ρ is well defined since it is the composition map of the usual symmetrization map from $\mathsf{S} = \mathrm{Sym}(V^*)$ into $T(V^*)$ with the canonical map from $T(V^*)$ onto $\mathsf{W} = T(V^*)/I$.

Using the symmetrization ρ, we pull back the product of W on S by:

$$F \underset{\rho}{\times} G := \rho^{-1}(\rho(F)\ \rho(G)), \quad \forall F, G \in \mathsf{S}.$$

We fix the value $t = 1$ of the parameter, and denote by $\star$ the $\underset{1}{\star}$-Moyal product.

PROPOSITION 1.8. *One has* $\underset{\rho}{\times} = \star$. *Therefore* $F \star G = \rho^{-1}(\rho(F)\ \rho(G))$, *for all* $F, G \in \mathsf{S}$ *and the algebra* $(\mathsf{S}, \star)$ *is isomorphic to the Weyl algebra* W.

Proof. It is clear from the definition that $\varphi^{\underset{\rho}{\times} k} = \varphi^k$ for all $\varphi \in V^*$, $k \in \mathbb{N}$. As in the proof of Lemma 1.6, one can write

$$\varphi_1 \dots \varphi_k = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varphi_{\sigma(1)} \underset{\rho}{\times} \cdots \underset{\rho}{\times} \varphi_{\sigma(k)}$$

and using the same Lemma once more, one has $\varphi^{\underset{\rho}{\times} k} = \varphi^{\star k}$, $\forall \varphi \in V^*$, $k \in \mathbb{N}$, and $\varphi_1 \underset{\rho}{\times} \cdots \underset{\rho}{\times} \varphi_k = \varphi_1 \star \cdots \star \varphi_k$, $\forall \varphi_1, \dots, \varphi_k \in V^*$. It results that $(\mathsf{S}, \underset{\rho}{\times})$ is generated by V^* as an algebra and the same holds for $(\mathsf{S}, \star)$ thanks to Remark 1.7. So in order to prove that $\underset{\rho}{\times} = \star$, we need only to prove that they do coincide on the linear generators of S, i.e. on the monomials $\varphi_1 \underset{\rho}{\times} \cdots \underset{\rho}{\times} \varphi_k = \varphi_1 \star \cdots \star \varphi_k$, for all $\varphi_1, \dots, \varphi_k \in V^*$. But this is trivial. □

As a consequence of Proposition 1.8, we can identify the Weyl algebra W with the Moyal algebra S endowed with the $\star$-product. Such an identification reveals itself to be useful, since the Moyal $\star$-product provides an explicit formula for the product of the Weyl algebra, i.e. for the product of polynomial differential operators. To illustrate this point, when $n = 1$ one has the following explicit formula:

$$F \underset{t}{\star} G = \sum_{k \geqslant 0} \frac{t^k}{2^k k!} \left(\sum_{r+s=k} (-1)^s \binom{k}{s} \frac{\partial^k F}{\partial p_1^r \partial q_1^s} \frac{\partial^k G}{\partial p_1^s \partial q_1^r} \right).$$

Consider now the basis $\{q_1^i \star p_1^j \mid (i, j) \in \mathbb{N}^2\}$ of $\mathsf{W}(1)$. By an easy computation, one obtains the following expression in terms of classical orthogonal polynomials:

$$q_1^i \star p_1^j = \begin{cases} (-1)^j \dfrac{j!}{2^j} L_j^{(i-j)}(2p_1q_1)\ q_1^{i-j}, & \text{if } i \geqslant j, \\ \\ (-1)^i \dfrac{i!}{2^i} L_i^{(j-i)}(2p_1q_1)\ p_1^{j-i}, & \text{if } i \leqslant j. \end{cases} \tag{5}$$

where $L_\beta^{(\alpha)}$ is the Laguerre polynomial (see e.g. [19]). Notice that (5) gives also the expression of the basis $\{q_1^{i_1} \star p_1^{j_1} \star \cdots \star q_n^{i_n} \star p_n^{j_n} \mid (i_1, j_1, \dots, i_n, j_n) \in \mathbb{N}^{2n}\}$ of $\mathsf{W}(n)$ since $q_1^{i_1} \star p_1^{j_1} \star \cdots \star q_n^{i_n} \star p_n^{j_n} = q_1^{i_1} \star p_1^{j_1} \cdots q_n^{i_n} \star p_n^{j_n}$.

Let us now define a filtration and also a $\mathbb{Z}_2$-grading of W. For the filtration, we keep the filtration of S, that is, we set:

$$\mathsf{W}_k := \bigoplus_{r \leqslant k} \mathsf{S}^r.$$

Using (2) and Proposition 1.4, it is clear that this is indeed a filtration of W and that the associated graded algebra is S.

What about the $\mathbb{Z}_2$-grading of W? It can be defined in two ways: first, consider W as the algebra of polynomial differential operators on $\mathbb{K}[y_1, \dots, y_n]$. Define an element of W to be *even* if it maps even polynomials into even polynomials and odd polynomials into odd ones. An *odd* element takes even polynomials into odd polynomials and vice versa. Evidently, this defines a $\mathbb{Z}_2$-grading on W and one has $\deg_{\mathbb{Z}_2}(p_1^{r_1} \star q_1^{s_1} \star \dots \star p_n^{r_n} \star q_n^{s_n}) \equiv (\sum_{i=1}^n r_i + \sum_{i=1}^n s_i) (\mathrm{mod}\ 2)$.

On the other hand, thanks to Proposition 1.4, one can define a $\mathbb{Z}_2$-grading on W by

$$\mathsf{W}_{\bar{0}} := \bigoplus_{k \geqslant 0} \mathsf{S}^{2k} \quad \text{and} \quad \mathsf{W}_{\bar{1}} := \bigoplus_{k \geqslant 0} \mathsf{S}^{2k+1}.$$

This $\mathbb{Z}_2$-grading is exactly the preceding one. Therefore W can be endowed with two Lie structures:

– a Lie algebra structure given by

$$[F, G]_{\mathscr{L}} := F \star G - G \star F, \quad \text{for all} F, G \in \mathsf{W}.$$

From (4), one has $[\mathsf{W}_k, \mathsf{W}_\ell]_{\mathscr{L}} \subset \mathsf{W}_{k+\ell-2}$, so W is a filtered Lie algebra for the shifted filtration $\mathsf{W}_k^{\mathscr{L}} := \mathsf{W}_{k+2}$ and its associated graded Lie algebra is S endowed with the Poisson bracket (using (4) once again). A widely known fact is that $[\mathsf{W}, \mathsf{W}]_{\mathscr{L}} = \mathsf{W}$ (we shall give a proof in Section 3), so there is no nonzero trace map on W satisfying $\mathrm{Tr}(F \star G) = \mathrm{Tr}(G \star F)$, for F, $G \in \mathsf{W}$, i.e. $H^1(\mathsf{W}) = \{0\}$ for the Lie algebra cohomology.

– a Lie superalgebra structure given by

$$[F, G] := F \star G - (-1)^{fg} G \star F, \quad \text{for all } F, G \in \mathsf{W}.$$

with $f = \deg_{\mathbb{Z}_2}(F)$ and $g = \deg_{\mathbb{Z}_2}(G)$.

A significant difference between the Lie algebra and the Lie superalgebra structures of W is set in the following Proposition:

PROPOSITION 1.9. *Let* Str *be the evaluation at* 0 *of the commutative algebra* S,

$$\mathrm{Str}(F) := F(0), \quad \textit{for all } F \in \mathsf{S}.$$

Then Str *is a supertrace on* W, *that is,* Str *is homogeneous and satisfies*

$$\mathrm{Str}([F, G]) = 0, \quad \textit{for all } F, G \in \mathsf{W}.$$

Proof. From Proposition 1.4,

$$F \star G = \sum_{k=0}^{\min(f,g)} C_k(F, G)$$

with $C_k(F,G) \in \mathsf{S}^{f+g-2k}$ for $F \in \mathsf{S}^f$ and $G \in \mathsf{S}^g$. One has $C_k(F,G) \in \mathbb{K}$ if and only if $k = \frac{f+g}{2}$. Since k runs from 0 to $\min(f,g)$, one can deduce: either $f \neq g$ and then $\mathrm{Str}(F \star G) = 0$, or $f = g$ and $\mathrm{Str}(F \star G) = C_f(F,G)$. In this later case, if f is even, then by (3), $C_f(F,G) = C_f(G,F)$, so $\mathrm{Str}(F \star G) = \mathrm{Str}(G \star F)$. If f is odd, $C_f(F,G) = -C_f(G,F)$ (see (3)), so $\mathrm{Str}(F \star G) = -\mathrm{Str}(G \star F)$. Either way, $\mathrm{Str}([F,G]) = 0$. Finally, $\mathrm{Str}(F) = 0$ if F is odd, so Str is homogeneous. □

Remark 1.10. Assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The Moyal $\star$-product is clearly not restricted to live only on polynomial functions: obviously, assuming that t is a formal parameter, Definition 1.2 defines an associative deformation of $\mathscr{C}^\infty(V)$. All above properties are true, in the formal sense, thanks to the density of polynomials in $\mathscr{C}^\infty(V)$ (with its usual Fréchet topology, see e.g. [20]).

On the other hand, if $F,\ G \in \mathscr{S}(V)$ (fast decreasing smooth functions), one has $\int F \star G = \int F\ G$, so one can define a trace by $\mathrm{Tr}(F) = \int F$ and this has important consequences (see e.g. [5, 17, 18]).

2. Embedding $\mathfrak{sp}(2n)$ and $\mathfrak{osp}(1,2n)$ into W

In the sequel, we denote by $\mathfrak{g}$ the Lie superalgebra $\mathfrak{osp}(1,2n)$ and by $\mathfrak{g}_{\bar{0}}$ its even part, i.e. $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2n)$. The Weyl algebra W is endowed with the super bracket $[F,G] = F \star G - (-1)^{fg} G \star F$, for $F \in \mathsf{S}^f$, $G \in \mathsf{S}^g$. Denote by ad the corresponding adjoint representation.

There exists a very well-known embedding of $\mathfrak{g}_{\bar{0}}$ into W given by:

PROPOSITION 2.1. *Given* $X \in \mathsf{S}^2$, *denote by* $\mathrm{ad}_{\mathsf{P}}(X) := \{X, .\}$. *Then*

$$\mathrm{ad}_{\mathsf{P}}(X)|_{\mathsf{S}^1} = \mathrm{ad}(X)|_{\mathsf{S}^1} \in \mathfrak{sp}(2n).$$

Moreover $[\mathsf{S}^2, \mathsf{S}^2] = \{\mathsf{S}^2, \mathsf{S}^2\} \subset \mathsf{S}^2$, *so that* S^2 *is a Lie algebra for the Poisson bracket or the* $\star$*-bracket and the map*

$$X \mapsto \mathrm{ad}_{\mathsf{P}}(X)|_{\mathsf{S}^1}$$

is a Lie algebra isomorphism from S^2 *onto* $\mathfrak{sp}(2n)$.

Proof. Define a nondegenerate skew symmetric bilinear form Φ on S^1 by

$$\Phi(\varphi, \psi) = \tfrac{1}{2}\{\varphi, \psi\} = \tfrac{1}{2}[\varphi, \psi]_{\mathscr{L}},$$

for $\varphi,\ \psi \in \mathsf{S}^1 = V^*$. Notice that when $X \in \mathsf{S}^2$, one has $\mathrm{ad}_{\mathsf{P}}(X) = \mathrm{ad}(X)$ by Proposition 1.5. Then $[\mathsf{S}^2, \mathsf{S}^2] = \{\mathsf{S}^2, \mathsf{S}^2\} \subset \mathsf{S}^2$ which implies that S^2 is a Lie algebra for the Poisson bracket (= $\star$-bracket up to "t"). Since $\mathrm{ad}_{\mathsf{P}}(X)$ is a derivation of the Poisson bracket, the bilinear form Φ is invariant and that means $\mathrm{ad}_{\mathsf{P}}(X)|_{\mathsf{S}^1} \in \mathfrak{sp}(2n)$. Let θ be the map from S^2 to $\mathfrak{sp}(2n)$ defined as $\theta(X) = \mathrm{ad}_{\mathsf{P}}(X)|_{\mathsf{S}^1}$. This map θ is clearly a Lie algebra homomorphism, which is injective since $\mathrm{ad}_{\mathsf{P}}(X)|_{\mathsf{S}^1} = 0$ implies $X \in \mathbb{K}$, thus $X = 0$. Also θ is onto since $\dim(\mathsf{S}^2) = \dim(\mathfrak{sp}(2n))$. □

In the remainder of the paper, we shall identify $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2n)$ with S^2 via the isomorphism of Proposition 2.1.

Let us define an embedding of $\mathfrak{g}$ into the Weyl algebra W by extending the previous embedding of $\mathfrak{g}_{\bar{0}}$ into W. This embedding has already been known for a long time among mathematical physicists (e.g. [7]) and it was used, for instance, to develop singleton Anti-de-Sitter theories [7]. This embedding was later described in [13] and [8]. We give the proof in [8]:

DEFINITION 2.2. The *twisted adjoint action* of the Lie superalgebra W on itself is defined by:

$$\mathrm{ad}'(F)(G) := F \star G - (-1)^{f(g+1)} G \star F, \quad \forall F \in \mathsf{S}^f, G \in \mathsf{S}^g.$$

Such a twisted action is typical of supersymmetry (see [2,9]). Remark that if $F \in \mathsf{W}_{\bar{0}}$, then $\mathrm{ad}'(F) = \mathrm{ad}(F)$.

PROPOSITION 2.3. *Let* $\mathfrak{s} = \mathsf{S}^1 \oplus \mathsf{S}^2$ *with* $\mathbb{Z}_2$*-grading induced by* W. *Then* $\mathfrak{s}$ *is a subalgebra of the Lie superalgebra* W. *Moreover* $\mathfrak{s} \simeq \mathfrak{osp}(1, 2n)$.

Proof. If $\varphi, \psi \in \mathsf{S}^1$, one has $\varphi \star \psi = \varphi\psi + \frac{1}{2}\{\varphi, \psi\}$, so $[\varphi, \psi] = 2\varphi\psi \in \mathsf{S}^2$. If $\varphi \in \mathsf{S}^1$ and $X \in \mathsf{S}^2$, one has $[\varphi, X] = \{\varphi, X\}$ (by Proposition 1.5), and $\{\varphi, X\} \in \mathsf{S}^1$. So $\mathfrak{s}$ is a subalgebra of the Lie superalgebra W.

Set $V = V_{\bar{0}} \oplus V_{\bar{1}}$ where $V_{\bar{0}} = \mathbb{K}$ and $V_{\bar{1}} = \mathsf{S}^1$. It is easy to check that $\mathrm{ad}'(\mathfrak{s})(V) \subset V$. For example, if $\varphi, \psi \in \mathsf{S}^1$, one has $\mathrm{ad}'(\varphi)(\psi) = [\varphi, \psi]_{\mathscr{L}} = \{\varphi, \psi\} \in \mathbb{K}$ and $\mathrm{ad}'(\varphi)(1) = 2\varphi$.

Define a bilinear form Θ on V by $\Theta(\varphi, \psi) = \frac{1}{2}\{\varphi, \psi\}$ and $\Theta(1, 1) = -1$. It is clear that Θ is a supersymmetric form, i.e. $\Theta(A, B) = (-1)^{ab}\Theta(B, A)$ for all $A \in V_a$ and $B \in V_b$. Straightforward computations show that Θ is ad'-invariant. Therefore we obtain a Lie superalgebra homomorphism φ that maps X to $\mathrm{ad}'(X)|_V$ from $\mathfrak{s}$ into $\mathfrak{osp}(1, 2n)$. Remark that $\mathrm{ad}'(X)|_V = 0$ for $X \in \mathsf{S}^1$ implies that $\mathrm{ad}'(X)(1) = 2X = 0$, so φ is injective and since $\dim(\mathfrak{s}) = \dim(\mathfrak{osp}(1, 2n))$, φ is an isomorphism. □

In the remainder of the paper, we shall identify

$$\mathfrak{g} = \mathfrak{osp}(1, 2n) \quad \text{with} \quad \mathfrak{s} = \mathsf{S}^1 \oplus \mathsf{S}^2$$

via the isomorphism given in Proposition 2.3. Then

$$\mathfrak{g} = \mathsf{S}^1 \oplus \mathsf{S}^2, \quad \mathfrak{g}_{\bar{1}} = \mathsf{S}^1 \quad \text{and} \quad \mathfrak{g}_{\bar{0}} = \mathsf{S}^2 = [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \mathfrak{sp}(2n).$$

The root system of $\mathfrak{g}$ is easily deduced: let

$$H_i = -\frac{1}{2}\,[p_i, q_i] \quad \text{for } 1 \leqslant i \leqslant n.$$

Set $\mathfrak{h} = \operatorname{span}\{H_1, \ldots, H_n\}$. It turns out that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}$. Let $\{\omega_1, \ldots, \omega_n\}$ be the dual basis. The root vectors and corresponding roots are:

- in $\mathfrak{g}_{\bar{1}}$:
  p_i with root ω_i (positive), q_i with root $-\omega_i$.
- in $\mathfrak{g}_{\bar{0}}$:
  $[p_i, q_j]$ $(i \neq j)$ with root $\omega_i - \omega_j$ (positive if $i < j$),
  $[p_i, p_j]$ with root $\omega_i + \omega_j$ (positive),
  $[q_i, q_j]$ with root $-(\omega_i + \omega_j)$.

The fundamental root system is $\{\omega_i - \omega_{i+1}\ (i = 1, \ldots, n-1), \omega_n\}$ with corresponding root vectors $[p_i, q_{i+1}]$ $(i = 1, \ldots, n-1)$ and p_n. These vectors generate the subalgebra $\mathfrak{n}^+$ of $\mathfrak{g}$ with basis the positive root vectors. Any simple finite-dimensional $\mathfrak{g}$-module has a *highest weight vector* v of *weight* $\lambda \in \mathfrak{h}^*$ satisfying $Hv = \lambda(H)v$ for all $H \in \mathfrak{h}$ and $\mathfrak{n}^+ v = 0$.

PROPOSITION 2.4. *The* $\mathfrak{g}$*-module* $\operatorname{ad}_{\mathsf{P}}(\mathfrak{g}_{\bar{0}})|_{\mathsf{S}^k}$ *is a simple module with highest weight vector* p_1^k *of weight* $k\omega_1$.

Proof. See [16] or [13] for a proof using Weyl's formula. We provide here a short direct proof: it is clear that $\operatorname{ad}_{\mathsf{P}}(\mathfrak{g}_{\bar{0}})(\mathsf{S}^k) \subset \mathsf{S}^k$. Now given $\varphi \in \mathsf{S}^1$, $\varphi \neq 0$, one can construct a Darboux basis with respect to Φ such that φ is the first basis vector. It results that there exists $A \in \operatorname{Sp}(2n)$ such that $\varphi = A(p_1)$. Denoting by A as well the corresponding isomorphism of the commutative algebra S, one has $\varphi^k = A(p_1^k)$. Since $\mathsf{S}^k = \operatorname{span}\{\varphi^k \mid \varphi \in \mathsf{S}^1\}$, one has $\operatorname{span}(\operatorname{Sp}(2n))(p_1^k) = \mathsf{S}^k$. One has a representation of $\operatorname{Sp}(2n)$ in S^k, $A \mapsto A|_{\mathsf{S}^k}$ with differential $\operatorname{ad}_{\mathsf{P}}(\mathfrak{g}_{\bar{0}})|_{\mathsf{S}^k}$. As a consequence, the $\mathfrak{g}_{\bar{0}}$-submodule of S^k generated by p_1^k is S^k itself. Since p_1^k is a highest weight vector of weight $k\omega_1$, S^k is a simple $\mathfrak{g}_{\bar{0}}$-module with highest weight $k\omega_1$. □

The next Proposition was proved in [15] for $\mathfrak{osp}(1,2)$ and in [13] for $\mathfrak{osp}(1,2n)$. We give a simplified proof that uses the Moyal $\star$-product.

PROPOSITION 2.5. *Let* $A_k = \mathsf{S}^{2k-1} \oplus \mathsf{S}^{2k}$, $k \geqslant 1$ *and* $A_0 = \{0\}$. *Then* A_k *is stable by* $\operatorname{ad}(\mathfrak{g})$. *Moreover,* A_k *is a simple* $\mathfrak{g}$*-module with highest weight vector* p_1^{2k} *of weight* $2k\omega_1$.

Proof. Let $X \in \mathfrak{g}_{\bar{1}}$ and $F \in \mathsf{S}^{2k}$. Then $\operatorname{ad}(X)(F) = \{X, F\}$ by Proposition 1.5, hence $\operatorname{ad}(X)(F) \in \mathsf{S}^{2k-1}$. Now assume that $F \in \mathsf{S}^{2k-1}$, then $\operatorname{ad}(X)(F) = X \star F + F \star X = 2XF \in \mathsf{S}^{2k}$. So A_k is $\operatorname{ad}(\mathfrak{g})$-stable. By Proposition 2.4, $A_k = \mathsf{S}^{2k-1} \oplus \mathsf{S}^{2k}$ is its decomposition into isotypic components under the action of $\operatorname{ad}(\mathfrak{g}_{\bar{0}})$. Any $\mathfrak{g}$-submodule U of A_k must be decomposed as $U = (U \cap \mathsf{S}^{2k}) \oplus (U \cap \mathsf{S}^{2k-1})$ and since S^{2k} and S^{2k-1} are simple $\mathfrak{g}_{\bar{0}}$-modules, one of them is contained in U if $U \neq \{0\}$ from which one deduces that $U = A_k$ using the beginning of the proof. Moreover, p_1^{2k} is clearly a highest vector of weight $2k\omega_1$. □

By completely similar arguments, we can decompose the twisted adjoint representation of $\mathfrak{g}$ in W:

PROPOSITION 2.6. *Let* $B_k = \mathsf{S}^{2k} \oplus \mathsf{S}^{2k+1}$, $k \geqslant 0$. *Then* B_k *is stable by* $\mathrm{ad}'(\mathfrak{g})$. *Moreover,* B_k *is a simple* $\mathfrak{g}$*-module with highest vector* p_1^{2k+1} *of weight* $(2k+1)\omega_1$.

Proof. Let $X \in \mathfrak{g}_{\bar{1}}$ and $F \in \mathsf{S}^{2k}$. Then $\mathrm{ad}'(X)(F) = 2XF \in \mathsf{S}^{2k}$ and if $F \in \mathsf{S}^{2k+1}$, then $\mathrm{ad}'(X)(F) = \{X, F\} \in \mathsf{S}^{2k}$, so B_k is $\mathrm{ad}'(\mathfrak{g})$-stable. By the same arguments in the proof of Proposition 2.5, one obtains easily that B_k is a simple $\mathfrak{g}$-module and p_1^{2k+1} is clearly a highest weight vector of weight $(2k+1)\omega_1$. □

Remark 2.7. From Propositions 2.5 and 2.6, it results that the only homomorphism of $\mathfrak{g}$-modules from $(\mathsf{W}, \mathrm{ad}(\mathfrak{g}))$ into $(\mathsf{W}, \mathrm{ad}'(\mathfrak{g}))$ (or the other way around) is zero.

3. Decomposition of the Adjoint and Twisted Adjoint W-Modules

Let us recall our conventions: the Weyl algebra W is a Lie algebra with bracket denoted by

$$[A, B]_{\mathscr{L}} = A \star B - B \star A, \quad \forall A, B \in \mathsf{W}.$$

Denote by $\mathrm{ad}_{\mathscr{L}}$ the corresponding adjoint representation. Thanks to the $\mathbb{Z}_2$-grading of the associative algebra W, there is a twisted adjoint representation of the Lie algebra W defined by:

$$\mathrm{ad}'_{\mathscr{L}}(A)(B) = A \star B - (-1)^a B \star A, \quad \forall A \in \mathsf{W}_a, B \in \mathsf{W}.$$

Note that $\mathrm{ad}'_{\mathscr{L}}(\mathsf{W}_{\bar{0}})(\mathsf{W}_i) \subset \mathsf{W}_i$ $(i = \bar{0}, \bar{1})$ and $\mathrm{ad}'_{\mathscr{L}}(\mathsf{W}_{\bar{1}})$ takes $\mathsf{W}_{\bar{0}}$ into $\mathsf{W}_{\bar{1}}$ and vice versa.

The Weyl algebra is also a Lie superalgebra with bracket denoted by

$$[A, B] = A \star B - (-1)^{ab} B \star A, \quad \forall A \in \mathsf{W}_a, B \in \mathsf{W}_b.$$

Denote by ad the corresponding adjoint representation. There is a twisted adjoint representation of the Lie superalgebra W defined by:

$$\mathrm{ad}'(A)(B) = A \star B - (-1)^{a(b+1)} B \star A, \quad \forall A \in \mathsf{W}_a, B \in \mathsf{W}_b.$$

This twisted adjoint representation was used in Proposition 2.4 to prove the embedding of $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ into W.

The decomposition of $\mathrm{ad}_{\mathscr{L}}(\mathfrak{g}_{\bar{0}}) = \mathrm{ad}'_{\mathscr{L}}(\mathfrak{g}_{\bar{0}})$ is given in Proposition 2.4, the decomposition of $\mathrm{ad}(\mathfrak{g})$ is given in Proposition 2.5 and the decomposition of $\mathrm{ad}'(\mathfrak{g})$ in Proposition 2.6.

We will now examine $\mathrm{ad}_{\mathscr{L}}$, $\mathrm{ad}'_{\mathscr{L}}$ as Lie algebra representations of W, ad and ad$'$ as Lie superalgebra representations of W. The main technical argument is given by the following theorem.

THEOREM 3.1. *Consider the* $\mathrm{ad} = \mathrm{ad}_{\mathscr{L}}$*-action of* $\mathfrak{g}_{\bar{0}}$ *on* W. *One has:*

1. *The map* $F \otimes G \mapsto F \star G$ *is a homomorphism of* $\mathfrak{g}_{\bar{0}}$*-modules from* $\mathsf{S}^\ell \otimes \mathsf{S}^m$ *onto* $\bigoplus_{k=0}^{\min(\ell,m)} \mathsf{S}^{\ell+m-2k}$, *so one has*
$$\mathsf{S}^\ell \star \mathsf{S}^m = \bigoplus_{k=0}^{\min(\ell,m)} \mathsf{S}^{\ell+m-2k}.$$
2. *The map* $F \otimes G \mapsto [F, G]_{\mathscr{L}}$ *is a homomorphism of* $\mathfrak{g}_{\bar{0}}$*-modules from* $\mathsf{S}^\ell \otimes \mathsf{S}^m$ *onto* $\bigoplus_{2p+1 \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-2(2p+1)}$, *so one has*
$$[\mathsf{S}^\ell, \mathsf{S}^m]_{\mathscr{L}} = \bigoplus_{2p+1 \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-2(2p+1)}.$$
3. *The map* $F \otimes G \mapsto [F, G]$ *is a homomorphism of* $\mathfrak{g}_{\bar{0}}$*-modules from* $\mathsf{S}^\ell \otimes \mathsf{S}^m$ *onto*
   - $\bigoplus_{2p+1 \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-2(2p+1)}$ *if* $\bar{\ell}\ \bar{m} \equiv \bar{0}$, *so in this case one has*
$$[\mathsf{S}^\ell, \mathsf{S}^m] = \bigoplus_{2p+1 \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-2(2p+1)}.$$
   - $\bigoplus_{2p+1 \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-4p}$ *if* $\bar{\ell}\ \bar{m} \equiv \bar{1}$, *so in this case one has*
$$[\mathsf{S}^\ell, \mathsf{S}^m] = \bigoplus_{2p \leqslant \min(\ell,m)} \mathsf{S}^{\ell+m-4p}.$$
4. *The map* C_k *is a homomorphism of* $\mathfrak{g}_{\bar{0}}$*-modules from* $\mathsf{S}^\ell \otimes \mathsf{S}^m$ *into* $\mathsf{S}^{\ell+m-2k}$, *so one has*
$$\begin{cases} C_k(\mathsf{S}^\ell, \mathsf{S}^m) = \mathsf{S}^{\ell+m-2k}, & \text{if } 0 \leqslant k \leqslant \min(\ell, m) \\ 0, & \text{otherwise.} \end{cases}$$

To prove the above Theorem, we need the following Lemma:

LEMMA 3.2. *Let* $\mathfrak{h}$ *be a Lie algebra,* $\mathscr{U} = \mathscr{U}(\mathfrak{h})$ *its enveloping algebra and* V *a semisimple* $\mathfrak{h}$*-module which decomposes as* $V = \bigoplus_{s=1}^r V_s$ *with* V_s *simple and pairwise two nonisomorphic. Let* $v = \sum_{s=1}^r v_s$ *with* $v_s \in V_s$ *and* $v_s \neq 0$. *Then* $V = \mathscr{U}(\mathfrak{h})v$.

Proof. The $\mathscr{U}$-module V is semisimple and $V = \bigoplus_{s=1}^r V_s$ is its decomposition into isotypic components. Let π_s be the projection of V onto V_s. Then π_s is an element of the bicommutant of the $\mathscr{U}$-module V and by the Jacobson density theorem, there exists $u \in \mathscr{U}$ such that $v_s = \pi_s(v) = uv$ and the result follows from the simplicity of each V_s. □

Proof of Theorem 3.1. For all $X \in \mathfrak{g}_{\bar{0}}$, $\mathfrak{g}_{\bar{0}}$ acts by $\mathrm{ad}(X)$, which is a derivation of the $\star$-product. One has $F \star G = \sum_{k=0}^{\min(\ell,m)} C_k(F, G)$, for all $F \in \mathsf{S}^\ell$, $G \in \mathsf{S}^m$ with $C_k(F, G) \in \mathsf{S}^{\ell+m-2k}$ which is irreducible under the $\mathrm{ad}(\mathfrak{g}_{\bar{0}})$-action. It results that the map $F \otimes G \mapsto F \star G$ is a homomorphism of $\mathfrak{g}_{\bar{0}}$-modules and so is any of the maps C_k.

If $L=\bigoplus_{k=0}^{\min(\ell,m)}\mathsf{S}^{\ell+m-2k}$, denote by ρ the map $\rho\colon \mathsf{S}^\ell\otimes\mathsf{S}^m\to L$, $\rho(F\otimes G)=F\star G$. We compute $\rho(p_1^\ell\otimes q_1^m)$: using (1), one has $\wp(p_1^\ell\otimes q_1^m)=\ell m p_1^{\ell-1}\otimes q_1^{m-1}$, so

$$\wp^k(p_1^\ell\otimes q_1^m)=\ell(\ell-1)\cdots(\ell-k+1)m(m-1)(m-k+1)p_1^{\ell-k}\otimes q_1^{m-k},$$

if $k\leqslant\min(\ell,m)$ and 0 otherwise. Then

$$C_k(p_1^\ell,q_1^m)=\frac{k!}{2^k}\binom{\ell}{k}\binom{m}{k}p_1^{\ell-k}q_1^{m-k},$$

if $k\leqslant\min(\ell,m)$ and 0 otherwise. Thus

$$\rho(p_1^\ell\otimes q_1^m)=\sum_{k=0}^{\min(\ell,m)}\frac{k!}{2^k}\binom{\ell}{k}\binom{m}{k}p_1^{\ell-k}q_1^{m-k}.$$

We can apply Lemma 3.2 to the $\mathfrak{g}_{\bar 0}$-submodule $\rho(\mathsf{S}^\ell\otimes\mathsf{S}^m)$ of L with $v=\rho(p_1^\ell\otimes q_1^m)$ to obtain that $\rho(\mathsf{S}^\ell\otimes\mathsf{S}^m)=L$.

Moreover C_k is a homomorphism of $\mathfrak{g}_{\bar 0}$-modules from $\mathsf{S}^\ell\otimes\mathsf{S}^m$ into $\mathsf{S}^{\ell+m-2k}$ and one has $C_k(\mathsf{S}^\ell,\mathsf{S}^m)=\{0\}$ if $k>\min(\ell,m)$. When $k\leqslant\min(\ell,m)$, $C_k(\mathsf{S}^\ell,\mathsf{S}^m)$ is a nonzero submodule of the simple module $\mathsf{S}^{\ell+m-2k}$ since $C_k(p_1^\ell,q_1^m)\neq 0$. Therefore $C_k(\mathsf{S}^\ell,\mathsf{S}^m)=\mathsf{S}^{\ell+m-2k}$.

The proof of (2) is completely similar, using (4). The same reasoning can also be used to prove (3): the definition of $[.,.]$ implies that $[F,G]=2\sum_{p\geqslant 0}C_{2p+1}(F,G)$ if $\overline{f}\overline{g}\equiv\overline{0}$ and $[F,G]=2\sum_{p\geqslant 0}C_{2p}(F,G)$ if $\overline{f}\overline{g}\equiv\overline{1}$. □

Remark 3.3.

1. Let us consider the case $n=1$, $\mathsf{W}=\mathsf{W}(1)$. Then $\mathfrak{g}_{\bar 0}=\mathfrak{sl}(2)$ and S^ℓ is the simple $(\ell+1)$-dimensional $\mathfrak{g}_{\bar 0}$-module, denoted by $D\left(\frac{\ell}{2}\right)$. Then $\dim(\mathsf{S}^\ell\otimes\mathsf{S}^m)=\dim\left(\bigoplus_{k=0}^{min(\ell,m)}\mathsf{S}^{\ell+m-2k}\right)$. So the map $F\otimes G\mapsto F\star G$ of Theorem 3.1(1) is an isomorphism from $D\left(\frac{\ell}{2}\right)\otimes D\left(\frac{m}{2}\right)$ onto

$$D\left(\frac{\ell}{2}+\frac{m}{2}\right)\oplus D\left(\frac{\ell}{2}+\frac{m}{2}-1\right)\oplus\cdots\oplus D\left(\left|\frac{\ell}{2}-\frac{m}{2}\right|\right),$$

providing an explicit and very handy formula for the computation of Clebsch–Gordan coefficients. Such a formula was used for instance in [1] to compute the commutation rules of some high-dimensional Lie algebras for which usual Clebsch–Gordan formulas were intractable. Unfortunately, it is easy to check that the $\star$-product will provide only a partial decomposition of the tensor product $\mathsf{S}^\ell\otimes\mathsf{S}^m$ when $n\geqslant 2$.

2. Using Theorem 3.1, one has the following identities:

$$\begin{aligned}&[\mathsf{S}^1,\mathsf{S}^k]_{\mathscr{L}}=\mathsf{S}^{k-1},\quad [\mathsf{S}^2,\mathsf{S}^k]_{\mathscr{L}}=\mathsf{S}^k\ (k\geqslant 1),\\ &[\mathsf{S}^3,\mathsf{S}^k]_{\mathscr{L}}=\mathsf{S}^{k-3}\oplus\mathsf{S}^{k+1}\ (k\geqslant 3),\\ &[\mathsf{S}^3,\mathsf{S}^2]_{\mathscr{L}}=\mathsf{S}^3\quad\text{and}\quad [\mathsf{S}^3,\mathsf{S}^1]_{\mathscr{L}}=\mathsf{S}^2.\end{aligned}\tag{6}$$

$$\begin{aligned}&[\mathsf{S}^1,\mathsf{S}^{2k}]=\mathsf{S}^{2k-1}\ (k\geqslant 1),\quad [\mathsf{S}^1,\mathsf{S}^{2k+1}]=\mathsf{S}^{2k+2}\ (k\geqslant 0),\\ &[\mathsf{S}^3,\mathsf{S}^{2k-1}]=\mathsf{S}^{2k-2}\oplus\mathsf{S}^{2k+2}\ (k\geqslant 2),\quad [\mathsf{S}^3,\mathsf{S}^1]=\mathsf{S}^4,\\ &[\mathsf{S}^3,\mathsf{S}^{2k}]=\mathsf{S}^{2k-3}\oplus\mathsf{S}^{2k+1}\ (k\geqslant 2)\quad\text{and}\quad [\mathsf{S}^3,\mathsf{S}^2]=\mathsf{S}^3.\end{aligned}\tag{7}$$

These identities turn out to be quite useful.

PROPOSITION 3.4. *One has*

$$\ker(\mathrm{Str})=[\mathfrak{g}_{\bar{0}},\mathsf{W}]=[\mathsf{W},\mathsf{W}]=\bigoplus_{k\geqslant 1}\mathsf{S}^k.$$

Proof. We need the following obvious result:

If $\mathfrak{h}$ *is a Lie algebra and* U *a nontrivial simple* $\mathfrak{h}$*-module, then* $\mathfrak{h}\,U=U$.

Applying this fact to $\mathfrak{g}_{\bar{0}}$ and the simple $\mathfrak{g}_{\bar{0}}$-module S^k ($k\geqslant 1$) we get that $[\mathfrak{g}_{\bar{0}},\mathsf{S}^k]=\mathsf{S}^k$ (alternatively, $[\mathsf{S}^2,\mathsf{S}^k]=\mathsf{S}^k$ by Formula (6)). As a consequence, $[\mathfrak{g}_{\bar{0}},\bigoplus_{k\geqslant 1}\mathsf{S}^k]=\bigoplus_{k\geqslant 1}\mathsf{S}^k$. By Proposition 1.9, one obtains $[\mathsf{W},\mathsf{W}]\subset\ker(\mathrm{Str})$, so finally we conclude that $\bigoplus_{k\geqslant 1}\mathsf{S}^k\subset[\mathsf{W},\mathsf{W}]\subset\ker(\mathrm{Str})$, but since $\mathrm{codim}\left(\bigoplus_{k\geqslant 1}\mathsf{S}^k\right)=\mathrm{codim}(\ker(\mathrm{Str}))=1$, the result is proved. □

COROLLARY 3.5 [13]. *One has* $\mathsf{W}=\mathbb{K}\oplus[\mathsf{W},\mathsf{W}]$.

Remark 3.6. The use of the supertrace Str is quite natural and provides a real simplification of Musson's proof, which does not use Str. Moreover, it sheds new light on the origin of the result.

THEOREM 3.7.

1. *The representation* $(\mathrm{ad}_{\mathscr{L}},\mathsf{W})$ *of the Lie algebra* W *is indecomposable with Jordan–Hölder series* $\{0\}\subset\mathbb{K}\subset\mathsf{W}$ *and one has* $\mathrm{ad}_{\mathscr{L}}(\mathsf{W})(\mathsf{W})=\mathsf{W}$.
2. *The representation* $(\mathrm{ad}'_{\mathscr{L}},\mathsf{W})$ *of the Lie algebra* W *is indecomposable,* $[\mathsf{W},\mathsf{W}]$ *is a simple subrepresentation,* $\mathsf{W}/[\mathsf{W},\mathsf{W}]$ *is the trivial representation and there exists a nontrivial* $[\mathsf{W},\mathsf{W}]$*-valued cocycle* ξ *defined by* $\xi(F)=0$ *if* F *is even, and* $\xi(F)=2F$ *if* F *is odd.*
3. *The representation* $(\mathrm{ad},[\mathsf{W},\mathsf{W}])$ *of the Lie superalgebra* W *is simple. Moreover, the* $\mathrm{ad}(\mathsf{W})$*-module* W *decomposes as* $\mathsf{W}=\mathbb{K}\oplus[\mathsf{W},\mathsf{W}]$.
4. *The representation* $(\mathrm{ad}',\mathsf{W})$ *of the Lie superalgebra* W *is simple.*

Proof.

1. Let M be a nontrivial $\mathrm{ad}_{\mathscr{L}}(\mathsf{W})$-module. Then M can be decomposed into isotypic components under the $\mathrm{ad}_{\mathscr{L}}(\mathfrak{g}_{\bar{0}})$-action. By Proposition 2.4, there exists k_0 such that $\mathsf{S}^{k_0} \subset M$. If $k_0 > 0$, using (6) one has $[\mathsf{S}^1, \mathsf{S}^{k_0}]_{\mathscr{L}} = \mathsf{S}^{k_0-1} \subset M$ implying that $\bigoplus_{k \leqslant k_0} \mathsf{S}^k \subset M$. Using $\mathrm{ad}_{\mathscr{L}}(\mathsf{S}^3)$ and (6), one deduces that $\mathsf{S}^{k_0+1} \subset M$ and repeating the argument, that $M = \mathsf{W}$.

   If the only k such that $\mathsf{S}^k \subset M$ is 0, then $M = \mathsf{S}^0 = \mathbb{K}$. Therefore, there is exactly one nontrivial invariant subspace, namely $\mathbb{K}$. It results that $\mathsf{W}/\mathbb{K}$ is simple, that the representation $(\mathrm{ad}_{\mathscr{L}}, \mathsf{W})$ is indecomposable, with Jordan-Hf́7lder series $\{0\} \subset \mathbb{K} \subset \mathsf{W}$. Finally, notice that $\mathrm{ad}_{\mathscr{L}}(\mathsf{W})(\mathsf{W})$ is invariant, contains strictly $\mathbb{K}$, so $\mathrm{ad}_{\mathscr{L}}(\mathsf{W})(\mathsf{W}) = \mathsf{W}$.
2. We start by proving that $\mathrm{Str}(\mathrm{ad}'_{\mathscr{L}}(F)(G)) = 0$, $\forall F, G \in \mathsf{W}$. If F is even, or if F and G are odd, then $\mathrm{ad}'_{\mathscr{L}}(F)(G) = \mathrm{ad}(F)(G)$, so we apply Proposition 1.9. Now assume that $F \in \mathsf{S}^k$, k odd and $G \in \mathsf{S}^\ell$, ℓ even. Then $\mathrm{ad}'_{\mathscr{L}}(F)(G) = F \star G + G \star F = 2\sum_{2s \leqslant \min(k,\ell)} C_{2s}(F, G)$ and $C_{2s}(F, G) \in \mathsf{S}^{k+\ell-4s}$. But $k + \ell - 4s > 0$, so $\mathrm{ad}'_{\mathscr{L}}(F)(G) \in \bigoplus_{r>0} \mathsf{S}^r = \ker(\mathrm{Str})$.

   It results that $\mathrm{ad}'_{\mathscr{L}}(\mathsf{W})(\mathsf{W}) \subset \ker(\mathrm{Str}) = [\mathsf{W}, \mathsf{W}]$ and a fortiori, $[\mathsf{W}, \mathsf{W}]$ is stable. The quotient $\mathsf{W}/[\mathsf{W}, \mathsf{W}]$ is the trivial representation. If F is invariant under $\mathrm{ad}'_{\mathscr{L}}(\mathsf{W})$, it is invariant under $\mathrm{ad}_{\mathscr{L}}(\mathfrak{g}_{\bar{0}})$ and has to be a constant. But $\mathrm{ad}'_{\mathscr{L}}(G)(1) = 2G$, if G is odd, so $F = 0$. Therefore the extension

   $$0 \to [\mathsf{W}, \mathsf{W}] \to \mathsf{W} \to \mathbb{K} = \mathsf{W}/[\mathsf{W}, \mathsf{W}] \to 0$$

   is nontrivial with corresponding nontrivial $[\mathsf{W}, \mathsf{W}]$-valued cocycle ξ defined by:

   $$\xi(F) = \mathrm{ad}'_{\mathscr{L}}(F)(1) = \begin{cases} 0, & \text{if } F \text{ is even.} \\ 2F, & \text{if } F \text{ is odd.} \end{cases}.$$

   Assume now that M is a nonzero $\mathrm{ad}'_{\mathscr{L}}(\mathsf{W})$-invariant subspace. As in (1), there exists k_0 such that $\mathsf{S}^{k_0} \subset M$. Let $X \in \mathsf{S}^1$ and $F \in \mathsf{S}^{k_0}$. One has $\mathrm{ad}'_{\mathscr{L}}(X)(F) = 2XF$, so $\mathrm{ad}'_{\mathscr{L}}(\mathsf{S}^1)(\mathsf{S}^{k_0}) = \mathsf{S}^{k_0+1}$ and $\bigoplus_{k \geqslant k_0} \mathsf{S}^k \subset M$. Take $X \in \mathsf{S}^3$, then $\mathrm{ad}'_{\mathscr{L}}(X)(F) = 2XF + 2C_2(X, F)$. As in the proof of Theorem 3.2, it results that $\mathrm{ad}'_{\mathscr{L}}(\mathsf{S}^3)(\mathsf{S}^{k_0}) = \mathsf{S}^{k_0+3} \oplus \mathsf{S}^{k_0-1}$ if $k_0 > 1$, so S^{k_0-1} is contained in M and then $\bigoplus_{k \geqslant 1} \mathsf{S}^k \subset M$. Thus, either $M = \bigoplus_{k \geqslant 1} \mathsf{S}^k = [\mathsf{W}, \mathsf{W}]$ or $M = \mathsf{W}$.
3. Thanks to Proposition 3.4, it remains to show that the representation $(\mathrm{ad}, [\mathsf{W}, \mathsf{W}])$ is simple. This is an easy consequence of (7): let M be a nonzero invariant subspace in $[\mathsf{W}, \mathsf{W}]$. Since M is $\mathrm{ad}(\mathfrak{g})$-stable, it decomposes into the isotypic components of $[\mathsf{W}, \mathsf{W}] = \bigoplus_{k \geqslant 1} A_k$ with $A_k = \mathsf{S}^{2k-1} \oplus \mathsf{S}^{2k}$ (see Proposition 2.5). Hence

   $$M = \bigoplus_{\substack{k \\ A_k \subset M}} A_k.$$

   Take k_0 to be the smallest k in this decomposition. There are two cases:

– if $k_0 = 1$, then $\mathfrak{g} = A_1 \subset M$ and $[\mathfrak{g}_{\bar{0}}, \bigoplus_{k\geqslant 1} \mathsf{S}^k] = \bigoplus_{k\geqslant 1} \mathsf{S}^k \subset M$, one has $M = [\mathsf{W}, \mathsf{W}]$ by Proposition 3.4.
– if $k_0 > 1$, then $[\mathsf{S}^3, A_{k_0}] \subset M$. By (7), we deduce that S^{2k_0-3}, S^{2k_0-2}, S^{2k_0+1} and S^{2k_0+2} are contained in M. But it results that $A_{k_0-1} = \mathsf{S}^{2k_0-3} \oplus \mathsf{S}^{2k_0-2}$ is contained in M, and we reach a contradiction.

4. The proof of (4) is completely similar: one uses the decomposition of the representation $(\mathrm{ad}, \mathsf{W})$ into isotypic components, $\mathsf{W} = \bigoplus_{k\geqslant 0} B_k$ where $B_k = \mathsf{S}^{2k} \oplus \mathsf{S}^{2k+1}$ (see Proposition 2.6).

□

4. Invariant Bilinear Forms for the Adjoint and Twisted Adjoint Actions of W

There is a bilinear form κ canonically associated to the supertrace on the Weyl algebra W, namely:

$$\kappa(F, G) := \mathrm{Str}(F \star G), \quad \forall F, G \in \mathsf{W}.$$

By Proposition 1.9, κ is supersymmetric:

$$\kappa(G, F) = (-1)^{fg}\kappa(F, G), \quad \forall F \in \mathsf{S}^f, G \in \mathsf{S}^g.$$

and from its very definition, κ is invariant under the adjoint representation:

$$\kappa(\mathrm{ad}(F)(G), H) + (-1)^{fg}\kappa(G, \mathrm{ad}(F)(H)) = 0, \quad \forall F \in \mathsf{S}^f, G \in \mathsf{S}^g.$$

Now, κ would be really interesting if it is nondegenerate, and this is indeed the case:

THEOREM 4.1. *The bilinear form κ on the Lie superalgebra W is nondegenerate, supersymmetric and* ad*-invariant. Moreover,*

$$\kappa(\mathsf{S}^\ell, \mathsf{S}^m) = \{0\}, \ \text{if } \ell \neq m,$$

and the restriction of κ to S^ℓ is nondegenerate. If κ' is another ad*-invariant bilinear form on W, there exists $\alpha \in \mathbb{K}$ such that $\kappa' = \alpha\ \kappa$ on* $[\mathsf{W}, \mathsf{W}]$.

Proof. Given $F \in \mathsf{S}^\ell, G \in \mathsf{S}^m$ with $\ell < m$, then $\kappa(F, G) = \mathrm{Str}\left(\sum_{k=0}^{\ell} C_k(F, G)\right)$ by Proposition 1.4, with $C_k(F, G) \in \mathsf{S}^{\ell+m-2k}$ and since $\ell + m - 2k > 0$, $\kappa(F, G) = 0$. Indeed, $\kappa(F, G) = 0$ if $\overline{\ell} + \overline{m} = \overline{1}$, since Str is homogeneous.

The form κ is nondegenerate, since its kernel is a two-sided ideal of W, which is a simple algebra [6]. One can also prove it directly as follows: we have to show that κ is nondegenerate on each component S^ℓ. Using Proposition 1.4, if F, $G \in \mathsf{S}^\ell$, then $\kappa(F, G) = C_\ell(F, G)$. Define $\varphi\colon \mathsf{S}^\ell \to (\mathsf{S}^\ell)^*$ as $\varphi(F)(G) = \kappa(F, G)$, for all F, $G \in \mathsf{S}^\ell$. Since κ is $\mathfrak{g}_{\bar{0}}$-invariant, φ is homomorphism from the $\mathfrak{g}_{\bar{0}}$-module S^ℓ into its contragredient module $(\mathsf{S}^\ell)^*$. Both are simple $\mathfrak{g}_{\bar{0}}$-modules and φ is nonzero since $C_\ell(\mathsf{S}^\ell, \mathsf{S}^\ell) = \mathsf{S}^0 = \mathbb{K}$ by Theorem 3.1(4), so φ is an isomorphism by Schur's Lemma, and this proves that κ is nondegenerate.

Assume now that κ' is an ad-invariant bilinear form. Then the map $F \mapsto \kappa'|_F$ where $\kappa'_F(G) = \kappa(F, G)$ is a homomorphism of $\mathfrak{g}_{\bar{0}}$-modules from S^ℓ into $(\mathsf{S}^m)^*$. When $\ell \neq m$, S^ℓ and $(\mathsf{S}^m)^* \equiv \mathsf{S}^m$ are not isomorphic, so $\kappa'(\mathsf{S}^\ell, \mathsf{S}^m) = \{0\}$ by Schur's Lemma. If $\ell = m$, S^ℓ is a simple highest weight $\mathfrak{g}_{\bar{0}}$-module, so it is Schur irreducible, and it results that there is on S^ℓ only one invariant bilinear form, up to a scalar, and consequently there exists $\alpha_\ell \in \mathbb{K}$ such that $\kappa'|_{\mathsf{S}^\ell \times \mathsf{S}^\ell} = \alpha_\ell \kappa|_{\mathsf{S}^\ell \times \mathsf{S}^\ell}$. Next we want to prove that $\alpha_\ell = \alpha_{\ell+1}$ for all $\ell \geqslant 0$. First we prove that $\alpha_{2\ell-1} = \alpha_{2\ell}$ for all $\ell \geqslant 1$. Note that $\mathrm{ad}(p_1)(q_1^{2\ell}) = 2\ell q_1^{2\ell-1}$, then from $\kappa'(\mathrm{ad}(p_1)(q_1^{2\ell}), F) = -\kappa'(q_1^{2\ell}, \mathrm{ad}(p_1)(F))$, for all $F \in \mathsf{S}^{2\ell-1}$, we deduce that:

$$(\alpha_{2\ell-1} - \alpha_{2\ell})\kappa(\mathrm{ad}(p_1)(q_1^{2\ell}), F) = 0,$$

and since κ is nondegenerate on $\mathsf{S}^{2\ell-1} \times \mathsf{S}^{2\ell-1}$, one can conclude that $\alpha_{2\ell-1} = \alpha_{2\ell}$ for all $\ell \geqslant 1$.

We then show that $\alpha_{2\ell} = \alpha_{2\ell+1}$ for all $\ell \geqslant 1$. One has $\kappa'(\mathrm{ad}(p_1^3)(q_1^{2\ell}), F) = -\kappa'(q_1^{2\ell}, \mathrm{ad}(p_1^3)(F))$, for all $F \in \mathsf{S}^{2\ell+1}$ and that implies

$$(\alpha_{2\ell+1} - \alpha_{2\ell})\kappa(\mathrm{ad}(p_1^3)(q_1^{2\ell}), F) = 0,$$

for all $F \in \mathsf{S}^{2\ell+1}$. But $\mathrm{ad}(p_1^3)(q_1^{2\ell})$ has component $6\ell p_1^2 q_1^{2\ell-1}$ on $\mathsf{S}^{2\ell+1}$ so $\alpha_{2\ell} = \alpha_{2\ell+1}$ as wanted.

Now, starting from $\ell = 1$, we conclude that $\kappa' = \alpha_1 \kappa$ on $[\mathsf{W}, \mathsf{W}]$. □

COROLLARY 4.2. W *and* $[\mathsf{W}, \mathsf{W}]$ *are superquadratic Lie superalgebras.*

Remark 4.3. Consider the adjoint representation of $\mathfrak{g}$ in W. By Proposition 2.5, it decomposes into isotypic components $\mathsf{W} = \bigoplus_{k \geqslant 0} A_k$, with $A_0 = \mathbb{K}$, $A_k = \mathsf{S}^{2k-1} \oplus \mathsf{S}^{2k}$, $k \geqslant 1$. Each A_k has an explicit invariant supersymmetric bilinear form, namely $\kappa|_{A_k \times A_k}$, so the ad-representation of $\mathfrak{g}$ in A_k is valued in

$$\mathfrak{osp}(\mathsf{S}^{2k}, \mathsf{S}^{2k-1}) = \mathfrak{osp}\left(\binom{2n+2k-1}{2k}, \binom{2n+2k-2}{2k-1}\right).$$

Also the ad-representation of $\mathfrak{g}_{\bar{0}}$ in S^k is either orthogonal or symplectic, according to the parity of k: S^{2k} is orthogonal, S^{2k+1} is symplectic, and the corresponding bilinear form is explicitly computable, since $\kappa|_{\mathsf{S}^k \times \mathsf{S}^k} = C_k$.

What about the ad′-representation of W? In what follows, we shall prove that it has also a nondegenerate supersymmetric invariant bilinear form. Actually, this bilinear form extends the one defined on $\mathsf{S}^0 \oplus \mathsf{S}^1$ when embedding $\mathfrak{osp}(1, 2n)$ in W (see Proposition 2.4), so it is directly related to orthosymplectic supersymmetry.

THEOREM 4.4. *Let* $B(F, G) = (-1)^{fg+1}\kappa(F, G)$, *for all* $F \in \mathsf{S}^f$, $G \in \mathsf{S}^g$. *Then* B *is a nondegenerate supersymmetric bilinear form on* W. *Moreover,* $B(\mathsf{S}^\ell, \mathsf{S}^m) = \{0\}$, *if* $\ell \neq m$, $B_{\mathsf{S}^\ell \times \mathsf{S}^\ell}$ *is nondegenerate and* B *is invariant under the* ad′*-representation of* W. *If* B' *is an* ad′*-invariant bilinear form on* W, *there exists* $\beta \in \mathbb{K}$ *such that* $B' = \beta\, B$.

Proof. It is easy to check that B is supersymmetric. Let us prove that B is ad$'$-invariant: consider $I = B(\mathrm{ad}'(A)(F), G) + (-1)^{af} B(F, \mathrm{ad}'(A)(G))$ with $\deg_{\mathbb{Z}_2}(A) = a$, $\deg_{\mathbb{Z}_2}(F) = f$ and $\deg_{\mathbb{Z}_2}(G) = g$. If $f + g + a = \overline{1}$, then $I = 0$. If $f + g + a = \overline{0}$, then

$$
\begin{aligned}
I &= (-1)^{(f+a)g+1}\kappa(\mathrm{ad}'(A)(F), G) + (-1)^{af}(-1)^{f(a+g)+1}\kappa(F, \mathrm{ad}'(A)(G)) \\
&= (-1)^{f}\Big((-1)^{a+1}\kappa(\mathrm{ad}'(A)(F), G) + (-1)^{af+1}\kappa(F, \mathrm{ad}'(A)(G))\Big) \\
&= (-1)^{f}\,\mathrm{Str}\Big((-1)^{a+1}(A \star F - (-1)^{a(f+1)}F \star A) \star G + \\
&\quad +(-1)^{af+1}F \star (A \star G - (-1)^{a(g+1)}G \star A)\Big) \\
&= (-1)^{f+a+1}\,\mathrm{Str}\Big(A \star F \star G - (-1)^{a(f+g)}F \star G \star A\Big) \\
&= (-1)^{f+a+1}\,\mathrm{Str}([A, F \star G]) = 0
\end{aligned}
$$

Since κ is nondegenerate, B is nondegenerate as well. Finally, in order to prove the uniqueness of two ad$'$-invariant bilinear forms modulo $\mathbb{K}$, one proceeds as in the proof of Theorem 4.1. □

5. Renormalized Supertrace and Formal Inverse Weyl Transform

In this section, we assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{P}$ be the algebra $\mathbb{K}[x_1, \ldots, x_n]$, $\mathscr{F}$ be the algebra $\mathbb{K}[[X_1, \ldots, X_n]]$, V be the space $V = \mathrm{span}\{X_1, \ldots, X_n\}$ and V^* be its dual, $V^* = \mathrm{span}\{x_1, \ldots, x_n\}$ with $\langle x_i, X_j\rangle = \delta_{ij}$.

There is a one-to-one mapping (the Laplace transform) from $\mathscr{P}^*$ onto $\mathscr{F}$ defined by the duality $\langle x^I | X^J\rangle = \delta_{IJ} I!$, where $x^I := x_1^{i_1}\cdots x_n^{i_n}$, $X^J := X_1^{j_1}\cdots X_n^{j_n}$, $I! := i_1!\cdots i_n!$. The spaces $\mathscr{P}^*$ and $\mathscr{F}$ can be identified and as a consequence, the Dirac distributions ∂_v, $v \in V$, $\partial_v(P) = P(v)$ become formal power series e^v so that:

$$P(v) = \langle P\,|\,\mathrm{e}^v\rangle, \ \ \forall\ P \in \mathscr{P}, v \in V \quad \text{(Taylor's Formula)}.$$

Also one has:

$$\left\langle \frac{\partial^I P}{\partial x^I}\,\middle|\, F\right\rangle = \langle P\,|\,X^I F\rangle, \quad \forall P \in \mathscr{P}, F \in \mathscr{F}$$

and this means that ${}^t\left(\frac{\partial^I}{\partial x^I}\right)$ is the multiplication by X^I in $\mathscr{F}$.

The algebra $\mathscr{P}$ has a Hopf algebra structure with coproduct $\Delta(P) := P(x + x')$ if one identifies $\mathscr{P} \otimes \mathscr{P}$ with $\mathbb{K}[x_1, \ldots, x_n, x'_1, \ldots, x'_n]$, and define the antipode $\mathscr{S}(P)(x) = P(-x)$.

Next we endow $\mathscr{P}$ with its natural topology, as defined in [4] and $\mathscr{F} = \mathscr{P}^*$ with the strong dual topology which is exactly the product topology $\Pi_{k\geqslant 0}\mathscr{F}^k$, where $\mathscr{F}^k$ denotes the set of homogeneous polynomials of degree k. Then the transposition map induces on $\mathscr{F}$ a topological Hopf algebra structure, which is exactly the usual structure (see [4]) with the identification $\mathscr{F}\widehat{\otimes}\mathscr{F} = \mathbb{K}[[X_1, \ldots, X_n, X'_1, \ldots, X'_n]]$ ($\widehat{\otimes}$ is the projective tensor product, see [20]).

Any linear operator $T: \mathscr{P} \to \mathscr{P}$ is continuous for the natural topology ([4]) and ${}^tT: \mathscr{F} \to \mathscr{F}$ is continuous. Denote by $\mathscr{L}(\mathscr{P})$ the space of linear operators on $\mathscr{P}$, and by $\mathscr{L}(\mathscr{F})$ the space of continuous linear operators on $\mathscr{F}$. Then one has:

$$\mathscr{L}(\mathscr{P}) = \mathscr{P}^* \widehat{\otimes} \mathscr{P} = \mathscr{P}^* \widehat{\otimes} \mathscr{P}^{**} = \mathscr{L}(\mathscr{F}) \quad \text{(see [20])}.$$

Let us quickly explain how it works: given $T \in \mathscr{L}(\mathscr{P})$, then $T = \sum_K \frac{1}{K!} P_K \otimes X^K$ with $P_K = T(x^K)$ and one has $T(P) = \sum_K \frac{1}{K!} P_K \langle X^K \mid P \rangle$ for all $P \in \mathscr{P}$. Now one has ${}^tT = \sum_K \frac{1}{K!} P_K \otimes X^K$ and

$${}^tT(F) = \sum_K \frac{1}{K!} \langle P_K \mid F \rangle X^K \text{ for all } F \in \mathscr{F}. \tag{8}$$

Given a polynomial map $d: V \to \mathscr{F}$ defined by $d(v) = \sum_K D_K(v) X^K$, $D_K \in \mathscr{P}$, there is an associated operator $D: \mathscr{P} \to \mathscr{P}$ defined by

$$D = \sum_K D_K(v) \frac{\partial^K}{\partial x^K}$$

and one has:

$$D(P)(v) = \langle P \mid d(v)\, \mathrm{e}^v \rangle, \quad \forall P \in \mathscr{P}, v \in V.$$

Since $\mathrm{span}\{\mathrm{e}^v \mid v \in V\}$ is dense in $\mathscr{F} = \mathscr{P}^*$ by the Hahn–Banach Theorem, we deduce:

$$\langle D(P) \mid F \rangle = \langle P \mid m \circ (d \otimes \mathrm{Id}) \circ \Delta(F) \rangle, \tag{9}$$

by noticing that $\Delta(\mathrm{e}^v) = \mathrm{e}^v \otimes \mathrm{e}^v$ and by extending d to $\mathscr{F}$ by $d(F) = \sum_K \langle D_K \mid F \rangle\ X^K$, $F \in \mathscr{F}$ (so $d(\mathrm{e}^v) = d(v)$).

The operator D is what we shall call a *differential operator* on $\mathscr{P}$. We shall say that D is a *differential operator of finite order* if $d(V) \subseteq \mathscr{F}_k = \{F \in \mathscr{F} \mid \deg(F) \leqslant k\}$, for some k.

A fundamental property of $\mathscr{P}$ is established by:

LEMMA 5.1 ([14]). *Any linear operator on $\mathscr{P}$ is a differential operator.*

Proof. Given $T \in \mathscr{L}(\mathscr{P})$, then ${}^tT \in \mathscr{L}(\mathscr{F})$, so we have to find a polynomial map d from V to $\mathscr{F}$ satisfying ${}^tT = m \circ (d \otimes \mathrm{Id}) \circ \Delta$ (see (9)) From the density of $\mathrm{span}\{\mathrm{e}^v \mid v \in V\}$, it is enough to prove the last identity on this set. Let $d = \sum_k D_K X^K$, $D_K \in \mathscr{P}$, then ${}^tT(\mathrm{e}^v) = (m \circ (d \otimes \mathrm{Id}) \circ \Delta)(\mathrm{e}^v) = \sum_K \langle D_k \mid \mathrm{e}^v \rangle X^K \mathrm{e}^v$ gives $\sum_K \langle D_K \mid \mathrm{e}^v \rangle X^K = {}^tT(\mathrm{e}^v)\, \mathrm{e}^{-v}$, hence $d(v) = {}^tT(\mathrm{e}^v)\, \mathrm{e}^{-v}$. □

Let us give an explicit formula: starting from $T \in \mathscr{L}(\mathscr{P})$, $T = \sum_I \frac{1}{I!} P_I \otimes X^I$ with $P_I \in \mathscr{P}$, then ${}^tT(\mathrm{e}^v) = \sum_I \frac{1}{I!} P_I(v) X^I$ by (8) so if $v = x_1 X_1 + \cdots + x_n X_n$ and $|I| = i_1 + \cdots + i_n$,

$$
\begin{aligned}
&{}^{t}T(\mathrm{e}^{v})\,\mathrm{e}^{-v}\\
&=\left(\sum_I \frac{1}{I!}P_I(v)X^I\right)\left(\sum_j (-1)^j\frac{v^j}{j!}\right)\\
&=\left(\sum_I \frac{1}{I!}P_I(v)X^I\right)\left(\sum_j \frac{(-1)^j}{j!}\sum_{i_1+\cdots+i_n=j}\frac{(i_1+\cdots+i_n)!}{i_1!\cdots i_n!}x_1^{i_1}\cdots x_n^{i_n}X_1^{i_1}\cdots X_n^{i_n}\right)\\
&=\left(\sum_I \frac{1}{I!}P_I(v)X^I\right)\left(\sum_I \frac{(-1)^{|I|}}{I!}x^I X^I\right)\\
&=\sum_N \frac{1}{N!}\left(\sum_{R+S=N}(-1)^{|S|}\frac{N!}{R!S!}P_R(v)x^S\right)X^N.
\end{aligned}
$$

Finally, $T=\sum_N \frac{1}{N!}\left(\sum_{R+S=N}(-1)^{|S|}\frac{N!}{R!S!}P_R x^S\right)\dfrac{\partial^N}{\partial x^N}$.

Recall that $P_R=T(x^R)$, therefore:

$$T=\sum_N \frac{1}{N!}\left(\sum_{R+S=N}(-1)^{|S|}\frac{N!}{R!S!}T(x^R)x^S\right)\frac{\partial^N}{\partial x^N}.$$

But $\Delta(x^N)=\sum_{R+S=N}\frac{N!}{R!S!}x^R\otimes x^S$, so this last formula leads us to:

THEOREM 5.2.

$$T=\sum_N \frac{1}{N!}\left(m\circ(T\otimes\mathscr{S})\circ\Delta(x^N)\right)\;\frac{\partial^N}{\partial x^N}\;,$$

where $\mathscr{S}$ is the antipode of $\mathscr{P}$.

For instance, when $n=1$, consider the operator T defined as $T(x^i)=x^j$ for fixed i and j, and 0 otherwise. Then;

$$T=\frac{x^j}{i!}\sum_{\ell\geqslant 0}(-1)^\ell\frac{x^\ell}{\ell!}\frac{\mathrm{d}^{i+\ell}}{\mathrm{d}x^{i+\ell}}.$$

Using the notation $P^I=p_1^{i_1}\cdots p_n^{i_n}$ and $Q^J=q_1^{j_1}\cdots q_n^{j_n}$, consider the (formal) completion $\overline{\mathsf{W}}=\mathbb{K}[Q][[P]]$ of the $\star$-algebra W. The Moyal product can be extended to $\overline{\mathsf{W}}$. Elements of $\overline{\mathsf{W}}$ are formal power series $\widetilde{F}=\sum_I \alpha_I(Q)\star P^I$ with $\alpha_I\in\mathbb{K}[Q]$. Define a map $\mathscr{W}:\overline{\mathsf{W}}\to\mathscr{L}(\mathscr{P})$ by:

$$\mathscr{W}(\widetilde{F})=\sum_I \alpha_I(x)\;\frac{\partial^I}{\partial x^I}.$$

Remark that $\mathscr{W}$ is simply the extension to the algebra $\overline{\mathsf{W}}$ of the natural W-module structure of $\mathscr{P}$ defined by $p_i\cdot F=\partial F/\partial x_i$, $q_i\cdot F=x_iF$, $i=1,\ldots,n$, hence $\mathscr{W}$ is an algebra homomorphism.

Lemma 5.1 can now be reinterpreted as:

PROPOSITION 5.3. *The map $\mathscr{W}$ is an isomorphism of algebras.*

It should be stressed that the domain of $\mathscr{W}$, i.e. $\overline{\mathsf{W}}$, is not at all identical to the formal completion $\overline{S}=\mathbb{K}[Q][[P]]$ endowed with an Abelian product and with elements $\widehat{F}=\sum_I \alpha_I(Q)\ P^I$.

Given $\widetilde{F}=\sum_I \alpha_I(Q)\star P^I\in\overline{\mathsf{W}}$, we can try to define its supertrace by using the supertrace of W that we will denote by $\mathrm{Str}_{\mathsf{W}}$. A natural candidate would be $\mathrm{Str}_{\overline{\mathsf{W}}}(\widetilde{F}):=\sum_I \mathrm{Str}_{\mathsf{W}}(\alpha_I(Q)\star P^I)$ but it is clear that this series happens to diverge. So $\mathrm{Str}_{\overline{\mathsf{W}}}$ has a domain denoted by $\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$. Evidently, $\mathsf{W}\subset\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$. On the other hand, $\mathscr{L}(\mathscr{P})$ is a $\mathbb{Z}_2$-graded algebra since $\mathscr{P}$ is $\mathbb{Z}_2$-graded, so we can define a supertrace at least on the ideal $\mathscr{L}_f(\mathscr{P})$ of finite-rank operators. Note that $\mathscr{L}_f(\mathscr{P})\cap\mathsf{W}=\{0\}$ since W is a simple algebra. So we now have two supertraces, living apparently on different domains, and we wish to compare these supertraces. This is done by the following theorem.

THEOREM 5.4.
One has $\mathscr{L}_f(\mathscr{P})\subset\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$ and if $\widetilde{F}\in\mathscr{L}_f(\mathscr{P})$,

$$\mathrm{Str}(\widetilde{F})=\frac{1}{2^n}\ \mathrm{Str}_{\overline{\mathsf{W}}}(\widetilde{F}).$$

To prove Theorem 5.4, we need explicit formulas for $\mathrm{Str}_{\mathsf{W}}$:

PROPOSITION 5.5.

1. *Using the natural isomorphism $\mathsf{W}(n)=\mathsf{W}(1)\otimes\cdots\otimes\mathsf{W}(1)$, one has*

$$\mathrm{Str}_{\mathsf{W}}(F_1\otimes\cdots\otimes F_n)=\mathrm{Str}_{\mathsf{W}}(F_1)\cdots\mathrm{Str}_{\mathsf{W}}(F_n).$$

2. *One has* $\mathrm{Str}_{\mathsf{W}}(P^I\star Q^J)=\delta_{IJ}\frac{I!}{2^{|I|}}$.

Proof.

1. Recall that the isomorphism $\mathsf{W}(n)=\mathsf{W}(1)\otimes\cdots\otimes\mathsf{W}(1)$ is defined by $F_1\otimes\cdots\otimes F_n=F_1\cdots F_n$. Then

$$\begin{aligned}\mathrm{Str}_{\mathsf{W}}(F_1\otimes\cdots\otimes F_n)&=\mathrm{Str}_{\mathsf{W}}(F_1\cdots F_n)=(F_1\cdots F_n)(0)=F_1(0)\cdots F_n(0)\\&=\mathrm{Str}_{\mathsf{W}}(F_1)\cdots\mathrm{Str}_{\mathsf{W}}(F_n).\end{aligned}$$

2. We start with the case $n=1$. Then $\mathrm{Str}_{\mathsf{W}}(p_1^{i_1}\star q_1^{j_1})=\kappa(p_1^{i_1},q_1^{j_1})=0$ if $i_1\neq j_1$ by Proposition 4.1. Furthermore, $\mathrm{Str}_{\mathsf{W}}(p_1^{i_1}\star q_1^{i_1})=C_{i_1}(p_1^{i_1},q_1^{i_1})=\frac{i_1!}{2^{i_1}}$, and it follows that:

$$\begin{aligned}\mathrm{Str}_{\mathsf{W}}(p_1^{i_1}\cdots p_n^{i_n}\star q_1^{j_1}\cdots q_n^{j_n})&=\mathrm{Str}_{\mathsf{W}}(p_1^{i_1}\star q_1^{j_1}\otimes p_2^{i_2}\star q_2^{j_2}\otimes\cdots\otimes p_n^{i_n}\star q_n^{j_n})\\&=\mathrm{Str}_{\mathsf{W}}(p_1^{i_1}\star q_1^{j_1})\cdots\mathrm{Str}_{\mathsf{W}}(p_n^{i_n}\star q_n^{j_n}),\end{aligned}$$

is equal to 0, if $I\neq J$, and to $\dfrac{I!}{2^{|I|}}$ if $I=J$. □

As a practical case of Theorem 5.4, we prove:

PROPOSITION 5.6. *Let T be the (finite-rank) operator defined by $T(x^I)=x^J$ for fixed I, J and* 0 *otherwise. Then*

$$T=\frac{x^J}{I!}\sum_S(-1)^S\frac{x^S}{S!}\frac{\partial^{I+S}}{\partial x^{I+S}},$$

and

$$\mathrm{Str}_{\overline{\mathsf{W}}}(T)=\begin{cases}2^n\ \mathrm{Str}(T)=2^n(-1)^{|I|}, & \textit{if } I=J\\ 0, & \textit{otherwise}\end{cases}.$$

Thus $T\in\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$ and its usual supertrace is, up to a factor $\frac{1}{2^n}$, its $\overline{\mathsf{W}}$-supertrace.

Proof. The formula for T is obtained by applying Theorem 5.2:

$$T=\sum_S(-1)^{|S|}\frac{Q^{J+S}}{I!S!}\star P^{I+S}.$$

Then $\mathrm{Str}_{\overline{\mathsf{W}}}(T)=\sum_S\frac{(-1)^{|S|}}{I!S!}\mathrm{Str}_{\mathsf{W}}(Q^{J+S}\star P^{I+S})=0.$, if $J\neq I$, by Proposition 5.5. When $I=J$, $\mathrm{Str}_{\overline{\mathsf{W}}}(Q^{I+S}\star P^{I+S})=(-1)^{|I+S|}\frac{(I+S)!}{2^{|I+S|}}$, so

$$\mathrm{Str}_{\overline{\mathsf{W}}}(T)=\frac{(-1)^{|I|}}{2^{|I|}}\sum_S\frac{(I+S)!}{I!S!}\frac{1}{2^{|S|}}.$$

If $|\tau|<1$, one has $\dfrac{1}{(1-\tau)^{i+1}}=\sum_s\binom{i+s}{s}\tau^s$, therefore:

$$\frac{1}{(1-\tau_1)^{i_1+1}\cdots(1-\tau_n)^{i_n+1}}=\sum_{s_1,\cdots,s_n}\binom{i_1+s_1}{s_1}\cdots\binom{i_n+s_n}{s_n}\tau_1^{s_1}\cdots\tau_n^{s_n},$$

if $|\tau_i|<1$, for all i. It results that:

$$\sum_S\frac{(I+S)!}{I!S!}\frac{1}{2^{|S|}}=\frac{1}{\left(1-\frac{1}{2}\right)^{i_1+1}\cdots\left(1-\frac{1}{2}\right)^{i_n+1}}=2^{|I|+n}.$$

At last, we obtain

$$\mathrm{Str}_{\overline{\mathsf{W}}}(T)=(-1)^{|I|}2^n=\mathrm{Str}(T)2^n.$$

□

Proof of Theorem 5.4. This proof is completely similar to the previous one and for this reason, we omit it (note that one can restrict to $T=\varphi\otimes x^K$, $\varphi\in\mathscr{P}^*$ since $\mathscr{L}_f(\mathscr{P})=\mathscr{P}^*\otimes\mathscr{P}$). □

Remark 5.7. Let us give some more interpretation about supertraces. We want to show how one can define a renormalized supertrace by using Theorem 5.4. We begin with $\mathscr{L}_f(\mathscr{P})$ and its natural supertrace Str. Notice that this supertrace is of intrinsic nature, since it is defined by

$$\mathrm{Str}(\varphi\otimes F)=(-1)^{\deg_{\mathbb{Z}_2}(\varphi)\deg_{\mathbb{Z}_2}(F)}\langle\varphi\mid F\rangle,\ \varphi\in\mathscr{P}^*,\ F\in\mathscr{P}.$$

Secondly, we have the supertrace $\mathrm{Str}_{\overline{\mathsf{W}}}$ defined on its domain $\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$ which contains W and $\mathscr{L}_f(\mathscr{P})$. On $\mathscr{L}_f(\mathscr{P})$, one has

$$\mathrm{Str}(T)=\frac{1}{2^n}\,\mathrm{Str}_{\overline{\mathsf{W}}}(T)$$

by Theorem 5.4. So we can extend Str to $\mathrm{Dom}(\mathrm{Str}_{\overline{\mathsf{W}}})$ and define a *renormalized supertrace*, denoted by RStr, by:

$$\mathrm{RStr}(T)=\frac{1}{2^n}\,\mathrm{Str}_{\overline{\mathsf{W}}}(T).$$

This extension is indeed a renormalized extension of Str: for instance, with the usual definition of the supertrace: $\mathrm{Str}(\mathrm{Id})=\infty-\infty$, a rather bad result, but with the renormalization:

$$\mathrm{RStr}(\mathrm{Id})=\frac{1}{2^n}.$$

Notice that $n=\dim(V)$ is the dimension of the underlying variety.

Next, we will make precise what we mean by a *formal inverse Weyl transform*. Recall that we can identify $\overline{\mathsf{W}}$ with $\mathscr{L}(\mathscr{P})$ thanks to Proposition 5.3. So, given $T\in\mathscr{L}(\mathscr{P})$, one can write:

$$T=\sum_I\alpha_I(Q)\star P^I\in\overline{\mathsf{W}}.$$

Consider the Abelian algebra $\mathbb{K}[[P,Q]]$, and denote by $\mathbb{K}[[P,Q]]^k$ the space of homogeneous polynomials of degree k. We endow $\mathbb{K}[[P,Q]]$ with the product topology $\mathbb{K}[[P,Q]]=\Pi_{k\geqslant 0}\mathbb{K}[[P,Q]]^k$ which is Fréchet. Now $\sum_I\alpha_I(Q)\star P^I\in\overline{\mathsf{W}}$ is a series (of polynomials) in $\mathbb{K}[[P,Q]]$, it converges in $\mathbb{K}[[P,Q]]$ if and only if for any k, the series obtained by taking the kth components converge in $\mathbb{K}[[P,Q]]^k$.

DEFINITION 5.8. When $\sum_I\alpha_I(Q)\star P^I$ converges in $\mathbb{K}[[P,Q]]$, we say that T has a *formal inverse Weyl transform* denoted by $\mathrm{IW}(T)$ and defined as the sum of the series.

Remark that taking $k=0$, we obtain that if T has a formal inverse Weyl transform, the series

$$\sum_I\left(\alpha_I(Q)\star P^I\right)(0)=\sum_I\mathrm{Str}_{\mathsf{W}}\left(\alpha_I(Q)\star P^I\right)=\mathrm{Str}_{\overline{\mathsf{W}}}(T)$$

must converge, so:

PROPOSITION 5.9. *The existence of* $\mathrm{RStr}(T)$ *is a necessary condition for the existence of the formal inverse Weyl transform of* T. *When* $\mathrm{IW}(T)$ *exists, one has:*

$$\mathrm{RStr}(T)=\frac{1}{2^n}\,\mathrm{IW}(T)(0).$$

As an operator from $\mathscr{L}(\mathscr{P})$ into $\mathbb{K}[[P,Q]]$, the formal inverse Weyl transform has a domain $\mathscr{D}\subsetneq\mathscr{L}(\mathscr{P})$ containing W and, naturally, it would be nice to have a characterization of $\mathscr{D}$. This will be done elsewhere, just let us develop here some examples. We take $n=1$ and consider the elementary operators E_{ij} on $\mathscr{P}$ defined by $E_{ij}(x^k)=\delta_{jk}x^i$. One has

$$E_{ij}=\frac{1}{j!}\sum_{\ell\geqslant 0}\frac{(-1)^\ell}{\ell!}q^{\ell+i}\star p^{\ell+j}$$

and some computation shows that

LEMMA 5.10. *The formal inverse Weyl transform of* E_{ij} *is*

$$\mathrm{IW}(E_{ij})=\begin{cases}(-1)^j2^{i-j+1}L_j^{(i-j)}(4pq)\,\mathrm{e}^{-2pq}\,q^{i-j}, & \text{if } j\leqslant i,\\ (-1)^i2^{j-i+1}\frac{i!}{j!}L_i^{(j-i)}(4pq)\,\mathrm{e}^{-2pq}\,p^{j-i}, & \text{if } j\geqslant i.\end{cases}$$

where $L_\beta^{(\alpha)}$ *is the Laguerre polynomial* (see [19]).

Consider now the operators S_λ defined by $S_\lambda(x^k)=\lambda^kx^k$, $\lambda\in\mathbb{K}$. Then by Theorem 5.2, one has:

$$S_\lambda=\sum_{\ell\geqslant 0}\frac{(-1)^\ell}{\ell!}(1-\lambda)^\ell q^\ell\star p^\ell.$$

By (5) in Section 1, $q^\ell\star p^\ell=(-1)^\ell\frac{\ell!}{2^\ell}L_\ell(2pq)$, so using the generating function of the Laguerre polynomials $L_\beta^{(0)}:=L_\beta$, one finds:

LEMMA 5.11. *The formal inverse Weyl transform of* S_λ *is*

$$\mathrm{IW}(S_\lambda)=\frac{2}{1+\lambda}\exp\left(2\frac{\lambda-1}{\lambda+1}pq\right),\quad \text{if } |1-\lambda|<2.$$

Notice that S_λ has a renormalized supertrace given by:

$$\mathrm{RStr}(S_\lambda)=\frac{1}{\lambda+1}.$$

When $|\lambda|<1$, this is exactly $\mathrm{Str}(S_\lambda)$.

Here are some cases of interest: first, take $\lambda=\mathrm{e}^\tau$, then $S_\lambda=\exp\left(\tau x\frac{\mathrm{d}}{\mathrm{d}x}\right)=\exp(\tau q\star p)$ and one finds that its formal inverse Weyl transform is:

$$\mathrm{IW}\left(\exp\left(\tau x\frac{\mathrm{d}}{\mathrm{d}x}\right)\right)=\frac{\mathrm{e}^{-\frac{\tau}{2}}}{\cosh\left(\frac{\tau}{2}\right)}\exp\left(2pq\tanh\left(\frac{\tau}{2}\right)\right),\quad \text{if } |1-\mathrm{e}^\tau|<2.$$

Take $\tau = i\theta$, $\theta \in \mathbb{R}$ with $\theta \neq (2s+1)\pi$, $s \in \mathbb{Z}$. Then $S_{e^{i\theta}} = \exp(i\theta q \star p)$ has formal inverse Weyl transform:

$$\mathrm{IW}\left(\exp\left(i\theta x \frac{\mathrm{d}}{\mathrm{d}x}\right)\right) = \frac{\mathrm{e}^{-i\frac{\theta}{2}}}{\cos\left(\frac{\theta}{2}\right)} \exp\left(2ipq \tan\left(\frac{\theta}{2}\right)\right),$$

and an interesting case is S_i with formal inverse Weyl transform $(1-i)\exp(2ipq)$ (see [3], where similar formulas are obtained, in the context of quantization of the harmonic oscillator). Note also that the value $\lambda = -1$ is critical: indeed S_{-1} is the parity operator, which has divergent renormalized supertrace and cannot have a formal inverse Weyl transform. So the estimation in (5.11) is the best one.

Let us now justify our construction somewhat. There is a natural question: quantum mechanics is built from operators on a Hilbert space, so where is the Hilbert space in that picture? Here is the answer: let $\mathscr{H}$ be the Hilbert space of entire functions $f(x)$, $x = ai + b$ such that $\int \mathrm{e}^{-|x|^2} |f(x)|^2 \,\mathrm{d}a\, \mathrm{d}b < \infty$ with scalar product $\langle f \mid g \rangle = \frac{1}{\pi} \int \mathrm{e}^{-|x|^2} f(x)\overline{g(x)} \,\mathrm{d}a\, \mathrm{d}b$. Then $\mathscr{P} \subset \mathscr{H}$, as a dense subspace and we can apply our algebraic formalism as follows: given a closed operator T on $\mathscr{H}$, whose domain contains $\mathscr{P}$, assume that $T(\mathscr{P}) \subset \mathscr{P}$ and denote by $T_{\mathscr{P}}$ the restriction of T to $\mathscr{P}$; to have a one-to-one correspondance $T \mapsto T_{\mathscr{P}}$, we have to assume more, for instance either T bounded, or $T_{\mathscr{P}}$ essentially self-adjoint, this is assumed in the foregoing. We can now define the formal inverse Weyl transform of T to be the formal inverse Weyl transform of $T_{\mathscr{P}}$ and the renormalized supertrace of T to be the renormalized supertrace of $T_{\mathscr{P}}$. For instance, consider S_i which is a unitary operator on $\mathscr{H}$, it has renormalized supertrace $\mathrm{RStr}(S_i) = \frac{1}{2}(1-i)$ and formal inverse Weyl transform $(1-i)\exp(2ipq)$. We are actually working in the so-called "coherent states formalism", and we can easily translate it in terms of the usual "metaplectic formalism". Define an operator $H\colon \mathscr{H} \to \mathscr{L}^2(\mathbb{R})$ by

$$H(f)(t) = \frac{\mathrm{e}^{-\frac{t^2}{2}}}{\pi^{\frac{3}{4}}} \int f(\sqrt{2}(t+i\xi))\, \mathrm{e}^{-\xi^2} \,\mathrm{d}\xi.$$

Let φ_n be the orthonormal basis of Hermite functions in $\mathscr{L}^2(\mathbb{R})$, and $Z_n = \frac{x^n}{\sqrt{n!}}$ the orthonormal basis of $\mathscr{H}$, then one has $H(Z_n) = \varphi_n$, $\forall n$, so H is a unitary isomorphism from $\mathscr{H}$ to $\mathscr{L}^2(\mathbb{R})$. The operator H maps the operator x (resp. $\frac{\mathrm{d}}{\mathrm{d}x}$) on $\mathscr{H}$ to the operator $\frac{1}{\sqrt{2}}\left(t - \frac{\mathrm{d}}{\mathrm{d}t}\right)$ (resp. $\frac{1}{\sqrt{2}}\left(t + \frac{\mathrm{d}}{\mathrm{d}t}\right)$) and we recover the usual metaplectic formalism (see [11] for details). Now we can explain our interest in S_i: indeed H maps S_i to the Fourier transform $\mathfrak{F}$ of $\mathscr{L}^2(\mathbb{R})$ and since "coherent states formalism" and "metaplectic formalism" are completely equivalent by H, we have a formal inverse Weyl transform of $\mathfrak{F}$, namely the formal inverse Weyl transform of S_i, i.e.

$$\mathrm{IW}(\mathfrak{F}) = (1-i)\exp(2ipq)$$

(some care must be given to the interpretation of p and q in terms of t), and also a renormalized supertrace

$$\mathrm{RStr}(\mathfrak{F}) = \frac{1}{1+i}.$$

So we think that our formalism might be of interest. Next step would be a Wigner's type formula: this is already done and will be explained in a subsequent paper.

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