THE AMITSUR-LEVITZKI THEOREM FOR THE ORTHOSYMPLECTIC LIE SUPERALGEBRA $\mathfrak{osp}(1,2n)$

PIERRE-ALEXANDRE GIÉ, GEORGES PINCZON, ROSANE USHIROBIRA

ABSTRACT. Based on Kostant's cohomological interpretation of the Amitsur-Levitzki theorem, we prove a super version of this theorem for the Lie superalgebras $\mathfrak{osp}(1,2n)$. We conjecture that no other classical Lie superalgebra can satisfy an Amitsur-Levitzki type super identity. We show several (super) identities for the standard super polynomials. Finally, a combinatorial conjecture on the standard skew supersymmetric polynomials is stated.

0. INTRODUCTION

In 1950, A. S. Amitsur and J. Levitzki proved the following theorem:

Theorem: *Given $X_1,\dots,X_k$ $n\times n$ complex matrices, define:*

$$S_k(X_1,\dots,X_k) := \sum_{\sigma\in\mathfrak{S}_k} \varepsilon(\sigma) X_{\sigma(1)}\dots X_{\sigma(k)}.$$

Then $S_{2n}(X_1,\dots,X_{2n}) = 0$ for all $X_1,\dots,X_{2n}$ $n\times n$ complex matrices.

The polynomial S_k is called the *standard polynomial* and we say that $\mathfrak{gl}(n)$ satisfies the standard polynomial identity of order $2n$. The polynomials S_k satisfy some recurrence relations and it is then easy to see that $S_k = 0$ if $k > 2n$. One can also check that $S_k \neq 0$ if $k < 2n$ (see Jacobson's book [8] for instance). Amitsur-Levitzki 's proof uses an inductive method that does not explain why such an identity exists [1, 8]. The Amitsur-Levitzki theorem gives a sort of measure of the "non-commutativity" of $\mathfrak{gl}(n)$, since if one defines the standard polynomial on an associative algebra A, it is clear that $S_2 = 0$ if A is commutative.

In his paper of 1958 [13], B. Kostant noted that the theorem above had been conjectured for some time by many mathematicians as I. Kaplansky, F. W. Levi and J. Levitzki himself. In this same paper, he gave a truly beautiful proof of the Amitsur-Levitzki theorem based on the cohomology of Lie algebras where the existence of the identity becomes clear.

Other proofs of the Amitsur-Levitzki theorem were later obtained: in 1963, R. G. Swan [24] reduced the original problem to a graph theory problem; in 1974, L. H. Rowen [21], used a direct method in his proof; in 1976, S. Rosset [20] gave a fast proof based on the Hamilton-Cayley theorem. Finally in 1981 [11], Kostant closed the subject once and for all by providing a very nice interpretation of the theorem in the context of representation theory and generalizing it using his separation of variables theorem [12]. To our knowledge, no one has returned to the Amitsur-Levitzki theorem since then.

In both Kostant's papers [13, 11], he was also concerned with a minimal index identity for the Lie subalgebras of $\mathfrak{gl}(n)$. He proved that for $\mathfrak{sl}(n)$, this index is $2n$ and for $\mathfrak{o}(2n+1)$, $4n+2$. He also showed that $\mathfrak{o}(2n)$ satisfies the standard polynomial identity of order $4n-2$ (as a consequence of the particular structure of its invariants due to the existence of the Pffafian), a result recovered by L. H. Rowen [22], but with some difficulties.

A few comments can be made about Kostant's proofs of the Amitsur-Levitzki theorem. First, both proofs use the polynomial structure of the ring of invariants of a semi simple Lie algebra. Second, his cohomological proof is based on a quite sophisticated theorem of cohomology of Lie algebras (namely, the Hopf-Koszul-Samelson theorem, see e.g. [10]) from which the Amitsur-Levitzki theorem is a consequence, modulo some combinatorial identities concerning the trace [13]. We can give a more economical proof based on similar arguments, but that does not rely on the Hopf-Koszul-Samelson theorem. Our proof uses only elementary

Mathematics Subject Classification (2000). 17B20, 17B56.
Key words. Lie superalgebras, Amitsur-Levitzki theorem, transgression operator.

properties of the Chevalley-Cartan's transgression operator [4, 3, 10] and some identities concerning the invariants $\mathrm{Tr}(X^k)$. It will not be presented in this paper; however, a completely similar reasoning will allow us to handle the orthosymplectic case $\mathfrak{osp}(1,2n)$.

The goal of this paper is to study possible versions of the Amitsur-Levitzki theorem in the case of Lie superalgebras. Consider the Lie superalgebra $\mathfrak{gl}(p,q)$ and define for $X_1,\dots,X_k \in \mathfrak{gl}(p,q)$:

$$\mathscr{A}_k(X_1,\dots,X_k) := \sum_{\sigma\in\mathfrak{S}_k} \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})X_{\sigma(1)}\dots X_{\sigma(k)}$$

where the *super sign* $\varepsilon(\sigma,\mathscr{X})$ will be defined in Section 1. The polynomial $\mathscr{A}_k$ is invariant under the action of the superalgebra $\mathfrak{gl}(p,q)$. We call $\mathscr{A}_k$ the *standard super (skewsymmetric) polynomial* of order k and it is clear that this polynomial is a natural candidate to replace S_k in the case of the superalgebra $\mathfrak{gl}(p,q)$. The next step is to check whether $\mathscr{A}_k$ is zero for k sufficiently big. However, if $pq \neq 0$, one can easily see that this is not true: there always exists a non nilpotent element $X \in \mathfrak{gl}(p,q)_{\bar{1}}$ and since $\mathscr{A}_k(X,\dots,X) = k!X^k$, it results that $\mathscr{A}_k \neq 0$ for all k. Therefore there is no standard super identity for $\mathfrak{gl}(p,q)$. With this counter-example in mind, one might think there is little hope in finding such an identity for the simple subalgebras of $\mathfrak{gl}(p,q)$.

Nevertheless, a closer look at the counter-example shows that it should be translated in terms of invariants: the algebra of invariants of $\mathfrak{gl}(p,q)$ is not finitely generated [23, 9]. If we follow the philosophy of Kostant's proofs, the algebra of invariants of the considered Lie superalgebra should be a polynomial algebra, which leaves us with a single choice: $\mathfrak{osp}(1,2n)$. For this series of Lie superalgebras, the algebra of invariants is a polynomial algebra in n variables (see [17, 23]). In addition, it is easy to see that all elements in $\mathfrak{osp}(1,2n)_{\bar{1}}$ are nilpotent (see Section 2), so the counter-example above does not apply.

As a consequence, the series $\mathfrak{osp}(1,2n)$ seems to be a good candidate for an Amitsur-Levitzki super theorem and our goal in this paper is to show that this super version does exists. The main result presented here is the following:

THEOREM: *For* $X_1,\dots,X_{4n+2} \in \mathfrak{osp}(1,2n)$, $\mathscr{A}_{4n+2}(X_1,\dots,X_{4n+2}) = 0$.

Notice that the number $4n+2$ appearing in the above theorem is precisely the one for $\mathfrak{gl}(2n+1)$ in the classical case of the Amitsur-Levitzki theorem.

The proof of this theorem follows the lines of Kostant's cohomological proof [13], but in a simpler form. We do not need to use a powerful theorem such as Hopf-Koszul-Samelson's for $\mathfrak{osp}(1,2n)$ (see [6]), but only elementary properties of a (super) transgression operator and some identities concerning super traces.

We believe that in general there is no super Amitsur-Levitzki theorem for the classical Lie superalgebras, with exception made to the series $\mathfrak{osp}(1,2n)$. This can be explained by the fact that their algebra of invariants is not (in general) finitely generated. Recall that $\mathfrak{osp}(1,2n)$ are the only simple Lie superalgebras (together with simple Lie algebras) that are also semi simple [5] (meaning complete reducibility of finite-dimensional representations). The fact that these Lie superalgebras satisfy an identity of Amitsur-Levitzki type strengthens the impression that they are very close to simple Lie algebras. However, the existence of a ghost center and of exotic primitive ideals in the enveloping algebra [18, 16, 17] indicate that the analogy cannot be carried much further.

Comments and Perspectives.

(1) We want to stress that the present study was performed in the context of the theory of invariants of Lie superalgebras and in that spirit. It would of course be interesting to relate our super identity with the general theory of PI-algebras (a very active domain, see e.g. [2, 19, 22]) where the classical Amitsur-Levitzki theorem plays an important role. That is a different study, which remains to be done since our super identity does not seem to appear in the PI-algebras literature.

(2) In Kostant [13], the standard polynomial for $\mathfrak{gl}(n)$ is translated in terms of the symmetric and alternating groups, and as a consequence it is proved that the Amitsur-Levitzki theorem implies a deep classical theorem of Frobenius. The super Amitsur-Levitzki identity does not hold for $\mathfrak{gl}(p,q)$. Nevertheless it would be very interesting to investigate how the canonical pairing between $GL(n)$ and the symmetric group can be extended to the super case and what, in that framework, would be the meaning of the standard super polynomial.

(3) We conjecture that a super Amitsur-Levitzki type identity holds not only for the canonical representation of $\mathfrak{osp}(1,2n)$ (as proved in the present paper), but also for any finite dimensional representation. In order to prove such a result, one should adapt the strategy developed in Kostant [11]. The Separation Theorem for $\mathfrak{osp}(1,2n)$ is needed, and it holds thanks to I. M. Musson [17]. But one also needs an analogue of Kostant's characterization of harmonics [12]: to our knowledge, this has never been done for $\mathfrak{osp}(1,2n)$ and is, in itself, a very interesting challenge.

Finally, we would like to point out that a succinct written version of this work, nearly without proofs, was published in [7].

1. Notations

1.1. **Algebras of supersymmetric and skew supersymmetric multilinear mappings.** Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a finite-dimensional $\mathbb{Z}_2$-graded vector space. Elements $X \in V$ are supposed homogeneous of degree $x \in \mathbb{Z}_2$. On $W = \mathbb{C}$, set $W_{\bar 0} = \mathbb{C}$ and $W_{\bar 1} = \{0\}$.

Let $\mathscr{F}(V,W)$ be the $\mathbb{Z}$-graded space of multilinear mappings from V into W, where W is a finite-dimensional $\mathbb{Z}_2$-vector space. Let $\mathscr{F}^p(V,W)$ be the subspace of p-linear mappings. One has $\mathscr{F}(V,W) = \oplus_{p\in\mathbb{Z}} \mathscr{F}^p(V,W)$ where $\mathscr{F}^0(V,W) = W$ and $\mathscr{F}^p(V,W) = \{0\}$ if $p \leq -1$. Consider the natural $\mathbb{Z}_2$-grading on $\mathscr{F}^p(V,W)$:

$$F \in \mathscr{F}^p(V,W),\ \deg_{\mathbb{Z}_2}(F) = f \text{ iff } \deg_{\mathbb{Z}_2}(F(X_1,\dots,X_p)) = x_1 + \cdots + x_p + f$$

The degree of $F \in \mathscr{F}^p(V,W)$ is denoted by $\deg(F) = (p,f)$ with $f = \deg_{\mathbb{Z}_2}(F)$ and in this case, we write $F \in \mathscr{F}^p_f(V,W)$.

Set $\mathscr{F}^p(V) := \mathscr{F}^p(V,\mathbb{C})$, for all $p \in \mathbb{N}$ and $\mathscr{F}(V) := \mathscr{F}(V,\mathbb{C})$.

For $\mathscr{X} = (X_1,\dots,X_p) \in V^p$ and σ an element in the symmetric group $\mathfrak{S}_p$, define the *super sign* :

$$\varepsilon(\sigma,\mathscr{X}) := (-1)^{K(\sigma,\mathscr{X})}$$

where $K(\sigma,\mathscr{X}) := \sharp\{(i,j) \mid i<j \text{ and } \sigma(i)>\sigma(j), X_{\sigma(i)}, X_{\sigma(j)} \in V_{\bar 1}\}$. It follows from the definition that $\varepsilon(\sigma,\mathscr{X})$ is a multiplier, that is:

$$\varepsilon(\sigma\sigma',\mathscr{X}) = \varepsilon(\sigma,\mathscr{X})\varepsilon(\sigma',\sigma^{-1}\cdot\mathscr{X})$$

with $\sigma\cdot\mathscr{X} := (X_{\sigma^{-1}(1)},\dots,X_{\sigma^{-1}(p)})$.

Besides the classical action, where $\sigma\cdot F(\mathscr{X}) = F(\sigma^{-1}\cdot\mathscr{X})$, this super sign allows us to consider two other actions of $\mathfrak{S}_p$ on $\mathscr{F}^p(V)$:

$$\begin{aligned} \sigma \underset{s}{\cdot} F(\mathscr{X}) &:= \varepsilon(\sigma,\mathscr{X})F(\sigma^{-1}\cdot\mathscr{X}) \\ \sigma \underset{a}{\cdot} F(\mathscr{X}) &:= \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})F(\sigma^{-1}\cdot\mathscr{X}) \end{aligned}$$

Say that a p-form F is *supersymmetric* if $\sigma \underset{s}{\cdot} F = F$, $\forall\ \sigma \in \mathfrak{S}_p$ and *skew supersymmetric* if $\sigma \underset{a}{\cdot} F = F$, $\forall\ \sigma \in \mathfrak{S}_p$.

Now let S and A be two operators on $\mathscr{F}(V)$ defined as:

$$\mathrm{S}(F) := \sum_{\sigma\in\mathfrak{S}_p} \sigma \underset{s}{\cdot} F, \quad \mathrm{A}(F) := \sum_{\sigma\in\mathfrak{S}_p} \sigma \underset{a}{\cdot} F, \quad \forall\, F \in \mathscr{F}^p(V).$$

These operators are actually projections and for $F \in \mathscr{F}(V)$, $\mathrm{S}(F)$ is supersymmetric and $\mathrm{A}(F)$ is skew supersymmetric.

We will denote by $\mathscr{P}(V)$ the space of supersymmetric forms and by $\mathscr{A}(V)$ the space of skew supersymmetric forms. These spaces are naturally $\mathbb{Z}_2$-graded and it is easy to see that $\mathscr{P}(V) = \mathrm{S}(\mathscr{F}(V))$ and $\mathscr{A}(V) = \mathrm{A}(\mathscr{F}(V))$.

Recall that $\mathscr{F}(V)$ is isomorphic to the dual of the tensor algebra $T(V)$ of V, which is isomorphic to the tensor algebra $T(V^*)$, providing for $\varphi_1,\dots,\varphi_p \in V^*$, $\deg_{\mathbb{Z}_2}(\varphi_i) = \phi_i$:

$$(\varphi_1\otimes\cdots\otimes\varphi_p)(X_1,\dots,X_p) := (-1)^{\Delta(\phi,x)}\varphi_1(X_1)\dots\varphi_p(X_p)$$

for all $X_1,\dots,X_p \in V$, $\deg_{\mathbb{Z}_2}(X_i) = x_i$, with $\phi = (\phi_1 \dots \phi_p)$, $x = (x_1 \dots x_p)$ and Δ being the 2-form with matrix $\begin{pmatrix} 0 & \dots & \dots & 0 \\ 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \dots & 1 & 0 \end{pmatrix}$.

Hence, the space $\mathscr{F}(V)$ is endowed with the usual tensor product $\otimes$, and also with a *super tensor product* denoted by $\underset{s}{\otimes}$ and defined as :

$$(F \underset{s}{\otimes} G)(X_1,\dots,X_{p+q}) := (-1)^{g(x_1+\dots+x_p)} F(X_1,\dots,X_p)G(X_{p+1},\dots,X_{p+q}),$$

for $X_1,\dots,X_{p+q} \in V$, $F \in \mathscr{F}^p(V)$ and $G \in \mathscr{F}^q_g(V)$. This product endows the space $\mathscr{F}(V)$ with a structure of a $\mathbb{Z}\times\mathbb{Z}_2$-graded associative algebra. Besides, for all $p \in \mathbb{N}^*$, elements $\varphi_1 \underset{s}{\otimes} \dots \underset{s}{\otimes} \varphi_p$ $(\varphi_i \in V^*)$ span $\mathscr{F}(V)$ as a space.

We will then define a product on $\mathscr{P}(V)$ and $\mathscr{A}(V)$:

$$F \cdot G := \frac{1}{p!q!} \mathrm{S}(F \underset{s}{\otimes} G),$$

for $F \in \mathscr{P}^p(V)$, $G \in \mathscr{P}^q(V)$,

$$F \wedge G := \frac{1}{p!q!} \mathrm{A}(F \underset{s}{\otimes} G),$$

for $F \in \mathscr{A}^p(V)$, $G \in \mathscr{A}^q(V)$.

To prove the associativity of these products we use the following technical Lemma:

Lemma 1.1. *Let $F \in \mathscr{F}^p(V)$, $G \in \mathscr{F}^q(V)$. Then $\mathrm{A}(F \underset{s}{\otimes} G) = \frac{1}{p!q!} \mathrm{A}(\mathrm{A}(F) \underset{s}{\otimes} \mathrm{A}(G))$ (same works for $\mathrm{S}(V)$).*

Proposition 1.2. *For the products defined above, $\mathscr{P}(V)$ is a $\mathbb{Z}_2$-graded associative algebra and $\mathscr{A}(V)$ is a $\mathbb{Z}\times\mathbb{Z}_2$-graded associative algebra. Moreover, for all $p \in \mathbb{N}^*$, elements $\varphi_1 \cdot \ldots \cdot \varphi_p$ $(\varphi_i \in V^*)$ span the space $\mathscr{P}(V)$ and elements $\varphi_1 \wedge \dots \wedge \varphi_p$ $(\varphi_i \in V^*)$ span the space $\mathscr{A}(V)$.*

Concerning commutation relations, we have:

Proposition 1.3. (a) *For $F \in \mathscr{P}_f(V)$, $G \in \mathscr{P}_g(V)$,*

$$F \cdot G = (-1)^{fg} G \cdot F\ .$$

(b) *For $F \in \mathscr{A}^p_f(V)$, $G \in \mathscr{A}^q_g(V)$,*

$$F \wedge G = (-1)^{pq+fg} G \wedge F.$$

Proof. Let $F \in V^*_f$. First of all, notice that by definition, $\deg_{\mathbb{Z}_2}(F(X)) = x + f$ for all $X \in V_x$. Hence $x + f \equiv 0$ mod 2 or $x + f \equiv 1$ mod 2. But $F(X) \in \mathbb{C} = \mathbb{C} \oplus \{0\}$. Therefore $x \equiv f$ mod 2 or $F(X) = 0$. In any case, $(-1)^x F(X) = (-1)^f F(X)$.

We will show the second assertion. Let $G \in V^*_g$. Then from the preceding remark;

$$\begin{aligned} (F \wedge G)(X,Y) &= (F \underset{s}{\otimes} G)(X,Y) - (-1)^{xy} (F \underset{s}{\otimes} G)(Y,X) \\ &= (-1)^{gx}\, F(X)G(Y) - (-1)^{xy}(-1)^{gy}\, F(Y)G(X) \\ &= (-1)^{gf}\, F(X)G(Y) - G(X)F(Y) \end{aligned}$$

and

$$\begin{aligned} (G \wedge F)(X,Y) &= (-1)^{fx}\, G(X)F(Y) - (-1)^{xy}(-1)^{fy}\, G(Y)F(X) \\ &= (-1)^{fg}\, G(X)F(Y) - F(X)G(Y). \end{aligned}$$

Therefore $F \wedge G = (-1)^{fg+1}\, G \wedge F$.

For arbitrary p and q, from the preceding Proposition, we can take $F = \varphi_1 \wedge \ldots \wedge \varphi_p$ and $G = \vartheta_1 \wedge \ldots \wedge \vartheta_q$ with $f = \phi_1 + \ldots + \phi_p$ and $g = \theta_1 + \ldots + \theta_q$. After successive commutations:

$$\begin{aligned} F \wedge G &= \varphi_1 \wedge \ldots \wedge \varphi_p \wedge \vartheta_1 \wedge \ldots \wedge \vartheta_q \\ &= (-1)^{\phi_p(\theta_1+\ldots+\theta_q)+q} \varphi_1 \wedge \ldots \wedge \varphi_{p-1} \wedge \vartheta_1 \wedge \ldots \wedge \vartheta_q \wedge \varphi_p \\ &= (-1)^{\phi_p g+q}(-1)^{\phi_{p-1}g+q} \varphi_1 \wedge \ldots \wedge \varphi_{p-2} \wedge \vartheta_1 \wedge \ldots \wedge \vartheta_q \wedge \varphi_{p-1} \wedge \varphi_p \\ &= \ldots \\ &= (-1)^{fg+pq}\, G \wedge F. \end{aligned}$$

This shows the second assertion. To prove the first assertion, it is enough to replace the skew supersymmetric action by the supersymmetric action. □

These relations imply that $\mathscr{P}(V)$ and $\mathscr{A}(V)$ are supercommutative with respect to their gradation. We can say that $\mathscr{P}(V)$ (respectively, $\mathscr{A}(V)$) is the analog of the algebra of polynomial functions (respectively, of the Grassmann algebra) in the non graded case:

Proposition 1.4. *There exists a superalgebra isomorphism between $\mathscr{P}(V)$ and $\mathrm{Sym}(V^*) = \mathrm{Sym}(V_0^*) \otimes \mathrm{Ext}(V_1^*)$ and an $\mathbb{Z} \times \mathbb{Z}_2$-graded superalgebra isomorphism between $\mathscr{A}(V)$ and $\mathrm{Ext}(V^*) = \mathrm{Ext}(V_{\bar 0}^*) \underset{\mathbb{Z}\times\mathbb{Z}_2}{\otimes} \mathrm{Sym}(V_{\bar 1}^*)$.*

Consider the following super bracket on $\mathrm{End}(\mathscr{A}(V))$:

$$[F,G] := F \circ G - (-1)^{pq+fg} G \circ F$$

with $\deg(F) = (p,f),\ \deg(G) = (q,g)$. We have then a $(\mathbb{Z} \times \mathbb{Z}_2)$-graded Lie superalgebra that will be denoted by $\mathfrak{gl}(\mathscr{A}(V))$.

Let D be a homogeneous element of degree (n,d) in the Lie superalgebra $\mathfrak{gl}(\mathscr{A}(V))$: that means $\deg(D(F)) = (p+n, f+d)$ for $f \in \mathscr{A}_f^p(V)$. We say that D is a *super derivation* of $\mathscr{A}(V)$ if for $F \in \mathscr{A}_f^p(V)$ and $G \in \mathscr{A}(V)$:

$$D(F \wedge G) = (DF) \wedge G + (-1)^{np+df} F \wedge (DG)$$

Proposition 1.5. *Let $X \in V_x$. Define $\mathrm{D}_X \in \mathfrak{gl}(\mathscr{P}(V))$ as:*

$$\mathrm{D}_X(F)(X_1,\ldots,X_{p-1}) := (-1)^{xf} F(X, X_1, \ldots, X_{p-1})$$

for $F \in \mathscr{P}_f^p(V)$. Then D_X is a super derivation of degree x of $\mathscr{P}(V)$.

Similarly,

Proposition 1.6. *Let $X \in V_x$. Define $\iota_X \in \mathfrak{gl}(\mathscr{A}(V))$ as:*

$$\iota_X(F)(X_1,\ldots,X_{p-1}) := (-1)^{xf} F(X, X_1, \ldots, X_{p-1})$$

for $F \in \mathscr{A}_f^p(V)$. Then ι_X is a super derivation of degree $(-1,x)$ of $\mathscr{A}(V)$.

1.2. **Cohomology of Lie superalgebras.** Let $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ be a Lie superalgebra with $\dim \mathfrak{g}_{\bar 0} = p$ and $\dim \mathfrak{g}_{\bar 1} = q$. The contragredient representation $\check{\mathrm{ad}}$ of the adjoint representation ad can be extended to a representation L^s of $\mathfrak{g}$ into $\mathscr{P}(\mathfrak{g})$ and to a representation L^a of $\mathfrak{g}$ into $\mathscr{A}(\mathfrak{g})$. For $X \in \mathfrak{g}$, L_X^s (resp. L_X^a) is the super derivation of degree x (resp. $(0,x)$) of $\mathscr{P}(\mathfrak{g})$ (resp. $\mathscr{A}(\mathfrak{g})$) defined as:

$$\mathrm{L}_X^{a,s} F(X_1,\ldots,X_n) := -(-1)^{xf} \sum_{j=1}^{n} (-1)^{x(x_1+\ldots+x_{j-1})} F(X_1,\ldots,\mathrm{ad}\,X(X_j),\ldots,X_n).$$

for $F \in \mathscr{P}_f^n(\mathfrak{g})$ (resp. $\mathscr{A}_f^n(\mathfrak{g})$). We denote by $I^s(\mathfrak{g})$ and $I^a(\mathfrak{g})$ the invariants under these actions.

Let d be the map from $\mathfrak{g}^*$ to $\mathscr{A}(\mathfrak{g})$ defined as:

$$d\phi(X_1,X_2) := -\phi([X_1,X_2]), \forall\ \phi \in \mathfrak{g}^*.$$

There exists a super derivation (also denoted by d) of $\mathscr{A}(\mathfrak{g})$ of degree $(1,0)$ extending d: for $F \in \mathscr{A}^n(\mathfrak{g})$,

$$dF(X_1,\ldots,X_{n+1}) := \sum_{i<j} (-1)^{i+j} \quad (-1)^{x_i(x_1+\ldots+x_{i-1})}(-1)^{x_j(x_1+\ldots+\widehat{x_i}+\ldots+x_{j-1})}$$
$$F([X_i,X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{n+1}).$$

From the Jacobi identity, we have $d^2 = 0$ and we can then define the cohomology (with trivial coefficients) of $\mathfrak{g}$ as:

$$Z(\mathfrak{g}) := \mathrm{Ker}(d),\ B(\mathfrak{g}) := \mathrm{Im}(d) \text{ and } H(\mathfrak{g}) := Z(\mathfrak{g})/B(\mathfrak{g}).$$

Let $\{X_1,\dots,X_{p+q}\}$ be a basis of $\mathfrak{g}$ and $\{\phi_1,\dots,\phi_{p+q}\}$ its dual basis. For $1 \le i \le p+q$, define the linear form $\widetilde{\phi_i}$ as $\widetilde{\phi_i}(X) := (-1)^{x_i x}\phi_i(X)$, $X \in \mathfrak{g}$. Thus, one has:

$$d = \frac{1}{2}\sum_{i=1}^{p+q} \widetilde{\phi_i} \wedge \mathrm{L}^a_{X_i}\,. \tag{1.1}$$

It is immediate from (1.1) that $I^a(\mathfrak{g}) \subset Z(\mathfrak{g})$. Moreover, one has:

$$\mathrm{L}^a_X = \iota_X \circ d + d \circ \iota_X,\ \forall\, X \in \mathfrak{g}. \tag{1.2}$$

As a consequence, L^a_X commutes with d and $\mathrm{L}^a_X(Z(\mathfrak{g})) \subset B(\mathfrak{g})$.

2. ORTHOSYMPLECTIC LIE SUPERALGEBRAS

In this section, let $\mathfrak{g}$ be the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2n)$. Among simple Lie superalgebras, the orthosymplectic $\mathfrak{osp}(1,2n)$ are the only ones (together with simple Lie algebras) satisfying the remarkable property of being semi simple [5], meaning that every finite-dimensional representation is completely reducible.

2.1. The Weyl algebra and $\mathfrak{osp}(1,2n)$.

In the quantization framework, the Lie superalgebra $\mathfrak{g}$ can be realized as follows: let A_n be the Weyl algebra generated by $\{p_i,q_i, i=1,\dots,n\}$ with $[p_i,q_i]_{\mathscr{L}} = 1$, $\forall\, i$, $[p_i,q_j]_{\mathscr{L}} = [p_i,p_j]_{\mathscr{L}} = [q_i,q_j]_{\mathscr{L}} = 0$, if $i \neq j$ where $[\cdot,\cdot]_{\mathscr{L}}$ denotes the Lie bracket. The algebra A_n is $\mathbb{Z}_2$-graded, hence a Lie superalgebra. Denote by $[\cdot,\cdot]$ its bracket.

The *twisted adjoint action* of A_n onto itself is defined as:

$$\mathrm{ad}' A(B) := AB - (-1)^{a(b+1)}BA$$

for $A,B \in \mathsf{A}_n$, $\deg_{\mathbb{Z}_2}(A) = a$, $\deg_{\mathbb{Z}_2}(B) = b$.

Let $V_{\bar{1}} := \mathrm{span}\{p_i,q_i, i=1,\dots,n\}$ and $\mathfrak{a} := V_{\bar{1}} \oplus [V_{\bar{1}},V_{\bar{1}}]$. Then $\mathfrak{a}$ is a subalgebra of the Lie superalgebra A_n. Let now $V := V_{\bar{0}} \oplus V_{\bar{1}}$ where $V_{\bar{0}} := \mathbb{C}\cdot 1$. We have $\mathrm{ad}'\,\mathfrak{a}(V) \subset V$. Moreover it is easy to see that the supersymmetric 2-form F defined on V as $F(X,Y) := [X,Y]_{\mathscr{L}}$, $F(X,1) = F(1,X) := 0$, for $X,Y \in V_{\bar{1}}$, and $F(1,1) := -2$ is $\mathrm{ad}'\,\mathfrak{h}$-invariant. It follows that $\mathfrak{h} \simeq \mathfrak{osp}(1,2n)$. An easy but remarkable consequence is the following

Proposition 2.1. *If $X \in \mathfrak{osp}(1,2n)_{\bar{1}}$, then $X^3 = 0$.*

Proof. It is enough to show that if $X \in V_{\bar{1}}$, then $(\mathrm{ad}' X|_V)^3 = 0$. Using $(\mathrm{ad}' X)(1) = 2X$ and $(\mathrm{ad}' X)^2(Y) = 2[X,Y]_{\mathscr{L}}\, X$, $\forall\, Y \in V_{\bar{1}}$, the result follows. □

More generally:

Proposition 2.2. *Let π be a finite-dimensional representation of $\mathfrak{g} = \mathfrak{osp}(1,2n)$. If $X \in \mathfrak{g}_{\bar{1}}$, then $\pi(X)$ is nilpotent.*

Proof. We use here the realization of $\mathfrak{g}$ as $\mathfrak{a}$. Let $X \in \mathfrak{a}_{\bar{1}} = V_{\bar{1}}$, $X \neq 0$. There exists a Darboux basis of $V_{\bar{1}}$ for the form $F|_{V_{\bar{1}}\times V_{\bar{1}}}$ such that X is the first basis element. We can then suppose that $X = p_1$. Let $\mathfrak{l} = \mathfrak{l}_{\bar{0}} \oplus \mathfrak{l}_{\bar{1}}$ with $\mathfrak{l}_{\bar{1}} = \mathrm{span}\{p_1,q_1\}$ and $\mathfrak{l}_{\bar{0}} = [\mathfrak{l}_{\bar{1}},\mathfrak{l}_{\bar{1}}]$. So $\mathfrak{l} \simeq \mathfrak{osp}(1,2)$. Let $\rho = \pi|_{\mathfrak{l}}$. Write $\rho = \oplus_{i\in I}\, \rho_i$ its decomposition into simple components. If $d_0 = \max\{\dim\rho_i, i \in I\}$, then $\pi(p_1)^{d_0} = 0$. □

2.2. Cohomology of $\mathfrak{osp}(1,2n)$.

From Djokovic-Hochschild [5], the representation L^a of $\mathfrak{g}$ into $\mathscr{A}(\mathfrak{g})$ is completely reducible. This fact and the results in Section 1.2, in particular the equations (1.1) and (1.2), allows us to prove

Lemma 2.3. *Every cohomology class of $H(\mathfrak{g})$ contains one and only one invariant cocycle. In particular,* 0 *is the unique invariant coboundary and $H(\mathfrak{g}) = I^a(\mathfrak{g})$.*

We omit the proof of this Lemma since it is very similar to the classical case that can be found in Koszul [14].

For the sake of completeness, we should mention that there exist better results concerning $H(\mathfrak{g})$: Fuks and Leites [6] have announced that $H(\mathfrak{g}) \simeq H(\mathfrak{g}_{\bar{0}}) = H(\mathfrak{sp}(2n))$. However, we shall not need these results here.

2.3. **Invariants.** It results from V. Kac's work that the Chevalley restriction theorem holds (see e.g. [17]): let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}_{\bar{0}}$ and W the Weyl group, then the restriction $\mathrm{Res}\colon I^s(\mathfrak{g}) \to J$ is an algebra isomorphism where $J := \mathrm{Sym}(\mathfrak{h}^*)^W$. As a consequence, $I^s(\mathfrak{g})$ is a polynomial algebra in n variables. We will now choose convenient generators. Let $\{\alpha_1, \dots, \alpha_n\}$ be the simple roots of $\mathfrak{g}$, so one has $\mathrm{Sym}(\mathfrak{h}^*) = \mathbb{C}[\alpha_1, \dots, \alpha_n]$. Set $t_k := \sum_{i=1}^n \alpha_i^{2k}$ for $k \in \mathbb{N}$, $k \geq 1$. The Weyl group W is generated by permutations and changes signs of $\alpha_1, \dots, \alpha_n$, so it is clear that $t_k \in J$, $\forall\, k$. Moreover, we can write $J = \mathbb{C}[t_1, \dots, t_n]$.

Lemma 2.4. *Let J_+ be the augmentation ideal of J. Then $t_k \in J_+^2$ if $k \geq n+1$.*

Proof. Let $k \geq n+1$. Since $t_k \in \mathbb{C}[t_1, \dots, t_n]$, there exists $\mathscr{I} \subset \mathbb{N}^n \setminus \{(0, \dots, 0)\}$ so that $t_k = \sum_{I \in \mathscr{I}} \lambda_I t_1^{i_1} \dots t_n^{i_n}$ with $\lambda_I \in \mathbb{C} \setminus \{0\}$. The monomial $t_1^{i_1} \dots t_n^{i_n}$ is homogeneous of degree $2(i_1 + 2i_2 + \dots + ni_n)$ for t_k is homogeneous of degree $2k$ (in the α_i's). Assume that $t_k \notin J_+^2$. There is no constant term in t_k, therefore there must exist $I = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 at the jth-position such that $\lambda_I \neq 0$. But then $\deg(t_k) = \deg(t_j)$ which implies $k = j$, a contradiction. Hence there is no I of such type in t_k, and we can conclude that $t_k \in J_+^2$. □

3. Chevalley's transgression operator for Lie superalgebras

The transgression operator $t\colon \mathrm{Sym}(\mathfrak{g}^*) \to \mathrm{Ext}(\mathfrak{g}^*)$ was introduced by Chevalley ([4, 3], see also [10]) and it is a fundamental tool in the theory of Lie algebras. In this section, we shall generalize this notion to the case of Lie superalgebras and give some elementary properties that will be useful in the sequel.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. Let $\{X_1, \dots, X_p\}$ be a basis of $\mathfrak{g}_{\bar{0}}$, $\{Y_1, \dots, Y_q\}$ a basis of $\mathfrak{g}_{\bar{1}}$, $\{\Omega_1, \dots, \Omega_p\}$ and $\{\phi_1, \dots, \phi_q\}$ their respective dual basis.

There exists an algebra homomorphism $s\colon \mathscr{P}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ such that $s(\Omega_i) = d\Omega_i$, $i = 1, \dots, p$, and $s(\phi_j) = d\phi_j$, $j = 1, \dots, q$ (since the $d\Omega_i$ commute, the $d\phi_j$ anticommute and the $d\Omega_i$, $d\phi_j$ commute, for $i = 1, \dots, p$, $j = 1, \dots, q$).

Note that from Proposition 1.4, $\{\Omega^I \cdot \phi^J \mid I \in \mathbb{Z}^p, J \in \mathbb{Z}_2^q\}$ is a basis for $\mathscr{P}(\mathfrak{g})$, where $\Omega^I = \Omega_1^{i_1} \cdot \dots \cdot \Omega_p^{i_p}$ and $\phi^J = \phi_1^{j_1} \cdot \dots \cdot \phi_q^{j_q}$.

Lemma 3.1. *One has $d(s(P)) = 0$, $\forall\, P \in \mathscr{P}(\mathfrak{g})$.*

Proof. Take a basis element $P = \Omega^I \cdot \phi^J = \Omega_1^{i_1} \cdot \dots \cdot \Omega_p^{i_p} \cdot \phi_1^{j_1} \cdot \dots \cdot \phi_q^{j_q} \in \mathscr{P}(\mathfrak{g})$. Since d is a super derivation and $d^2 = 0$ then:

$$d(s(P)) = d((d\Omega_1)^{i_1} \wedge \dots (d\Omega_p)^{i_p} \wedge (d\phi_1)^{j_1} \wedge \dots \wedge (d\phi_q)^{j_q}) = 0$$

□

There exists a super derivation R of $\mathscr{P}(\mathfrak{g})$ of degree 0 extending $\mathrm{Id}_{\mathfrak{g}^*}$:

$$\mathrm{R} := \sum_{i=1}^{p} \Omega_i \cdot \mathrm{D}_{X_i} - \sum_{j=1}^{q} \phi_j \cdot \mathrm{D}_{Y_j}$$

Recall $\mathscr{P}(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}_{\bar{0}}^*) \otimes \mathrm{Ext}(\mathfrak{g}_{\bar{1}}^*)$. Therefore for $P \in \mathscr{P}(\mathfrak{g})$, one can consider $\deg_{\mathbb{Z}}(P)$ from the natural $\mathbb{Z}$-gradations of $\mathrm{Sym}(\mathfrak{g}_{\bar{0}}^*)$ and $\mathrm{Ext}(\mathfrak{g}_{\bar{1}}^*)$.

Lemma 3.2. *One has $\mathrm{R}(P) = (\deg_{\mathbb{Z}}(P))\, P$, for all $P \in \mathscr{P}(\mathfrak{g})$.*

Proof. Remark that the super derivation D_{X_i} corresponds to the super derivation $\dfrac{\partial}{\partial \Omega_i}$ of $\mathrm{Sym}(\mathfrak{g}_{\bar{0}})$ (of degree $\bar{0}$) and the super derivation D_{Y_j} corresponds to the super derivation $\dfrac{\partial}{\partial \phi_j}$ of $\mathrm{Ext}(\mathfrak{g}_{\bar{1}})$ (of degree $\bar{1}$). Then the assertion follows easily. □

Following Chevalley, we now define the *transgression operator* $t\colon \mathscr{P}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ as

$$t(P) := \sum_{i=1}^{p} \Omega_i \wedge s(\mathrm{D}_{X_i}(P)) - \sum_{j=1}^{q} \phi_j \wedge s(\mathrm{D}_{Y_j}(P)),\ \forall\, P \in \mathscr{P}(\mathfrak{g}) \tag{3.1}$$

A priori, this definition seems to be basis dependent, but this is not the case as we show below. For now, let us state:

Lemma 3.3. *The operator t is an s-derivation, that is, $t(P\cdot Q) = t(P)\wedge s(Q) + s(P)\wedge t(Q)$, for all $P,Q \in \mathscr{P}(\mathfrak{g})$.*

Proof. Notice that D_{X_i} has degree $\overline{0}$ and D_{Y_j} has degree $\overline{1}$. By definition:

$$\begin{aligned} t(P\cdot Q) &= \sum_{i=1}^{p} \Omega_i \wedge s\left(\mathrm{D}_{X_i}(P\cdot Q)\right) - \sum_{j=1}^{q} \phi_j \wedge s\left(\mathrm{D}_{Y_j}(P\cdot Q)\right) \\ &= \sum_{i=1}^{p} \Omega_i \wedge \left[s\left(\mathrm{D}_{X_i}(P)\right)\wedge s(Q) + s(P)\wedge s\left(\mathrm{D}_{X_i}(Q)\right)\right] \\ &\quad - \sum_{j=1}^{q} \phi_j \wedge \left[s\left(\mathrm{D}_{Y_j}(P)\right)\wedge s(Q) + (-1)^{\deg_{\mathbb{Z}_2}(P)} s(P)\wedge \left(\mathrm{D}_{Y_j}(Q)\right)\right] \end{aligned}$$

If $P = \Omega^I\cdot\phi^J = \Omega_1^{i_1}\cdot\ldots\cdot\Omega_p^{i_p}\cdot\phi_1^{j_1}\cdot\ldots\cdot\phi_q^{j_q} \in \mathscr{P}(\mathfrak{g})$, then $\deg_{\mathbb{Z}_2}(P) = \overline{|J|}$ and $\deg_{\mathbb{Z}\times\mathbb{Z}_2}(s(P)) = (2(|I| + |J|), \overline{|J|})$. As a consequence, since $\deg_{\mathbb{Z}\times\mathbb{Z}_2}(\Omega_i) = (1,\overline{0})$ and $\deg_{\mathbb{Z}\times\mathbb{Z}_2}(\phi_j) = (1,\overline{1})$, we have

$$\Omega_i \wedge s(P) = s(P)\wedge\Omega_i \quad\text{ and } \phi_j\wedge s(P) = (-1)^{\deg_{\mathbb{Z}_2}(P)} s(P)\wedge\phi_j.$$

□

Lemma 3.4. *One has $d(t(P)) = s(\mathrm{R}(P))$, $\forall\, P \in \mathscr{P}(\mathfrak{g})$.*

Proof. It is enough to apply Lemma 3.1 and to use the fact that d has degree $\overline{1}$. □

Since $\mathrm{R}(P) = (\deg_{\mathbb{Z}}(P))\,P$, Lemma 3.4 shows that if P has no constant term, then $s(P)$ is a coboundary.

Lemma 3.5. *For all $P \in \mathrm{Sym}(\mathfrak{g}^*)$ and $X\in\mathfrak{g}$, $s(\mathrm{L}^s_X(P)) = \mathrm{L}^a_X(s(P))$.*

Proof. Let $P,Q \in \mathrm{Sym}(\mathfrak{g}^*)$ and $X\in\mathfrak{g}$. We have:

$$s(\mathrm{L}^s_X(P\cdot Q)) = s(\mathrm{L}^s_X(P))\wedge s(Q) + (-1)^{x\deg_{\mathbb{Z}_2}(P)} s(P)\wedge s(\mathrm{L}^s_X(Q))$$

and

$$\mathrm{L}^a_X(s(P\cdot Q)) = \mathrm{L}^a_X(s(P)\wedge s(Q)) = \mathrm{L}^a_X(s(P))\wedge s(Q) + (-1)^{x\deg_{\mathbb{Z}_2}(s(P))} s(P)\wedge\mathrm{L}^a_X(s(Q)).$$

Note that $\deg_{\mathbb{Z}_2}(P) = \deg_{\mathbb{Z}_2}(s(P))$. So it is enough to consider basis elements of $\mathrm{Sym}(\mathfrak{g}^*)$. Therefore from $s(\mathrm{L}^s_X(\Omega_i)) = d(\check{\mathrm{ad}}_X(\Omega_i))$, $\forall 1\le i\le p$ and $\mathrm{L}^a_X(s(\Omega_i)) = \mathrm{L}^a_X(d\Omega_i) = d(\mathrm{L}^a_X(\Omega_i)) = d(\check{\mathrm{ad}}_X(\Omega_i))$ (the operators L^a_X and d commute), the assertion follows. The same works for ϕ_i, $1\le i\le q$. □

Hence, s is a homomorphism of $\mathfrak{g}$-modules if $\mathscr{P}(\mathfrak{g})$ is endowed with the representation L^s and $\mathscr{A}(\mathfrak{g})$ endowed with the representation L^a. Therefore $s(I^s(\mathfrak{g}))\subset I^a(\mathfrak{g})$.

In order to establish other properties of the transgression, we need now an intrinsic definition of t. First, observe that there is an isomorphism $\mathrm{End}(\mathfrak{g}) = \mathfrak{g}^*\underset{s}{\otimes}\mathfrak{g}$ given by:

$$(\Omega\underset{s}{\otimes}X)(Y) := (-1)^{xy}\Omega(Y)\,X,\ \forall\,\Omega\in\mathfrak{g}^*,\ X,Y\in\mathfrak{g}.$$

Thanks to this identification, the representation $\pi := \check{\mathrm{ad}}\otimes\mathrm{ad}$ becomes $\mathrm{ad}(\mathrm{ad}\,\cdot)$ and $\mathrm{Id}_{\mathfrak{g}} = \sum_{i=1}^{p}\Omega_i\underset{s}{\otimes}X_i - \sum_{j=1}^{q}\phi_j\underset{s}{\otimes}Y_j$ is π-invariant.

Now fix $P\in\mathscr{P}(\mathfrak{g})$ and set $\tau_P\colon\ \mathrm{End}(\mathfrak{g})\to\mathscr{A}(\mathfrak{g})$ as

$$\tau_P(\Omega\underset{s}{\otimes}X) := \Omega\wedge s(\mathrm{D}_X(P)),\ \forall\Omega\in\mathfrak{g}^*, X\in\mathfrak{g}. \tag{3.2}$$

It is immediate that $\tau_P(\mathrm{Id}_{\mathfrak{g}}) = t(P)$, so the definition of t in (3.1) is basis independent. In addition, using the representation π on $\mathrm{End}(\mathfrak{g})$ and L^a on $\mathscr{A}(\mathfrak{g})$, one has

Lemma 3.6. *If $P \in I^s(\mathfrak{g})$, then $\tau_P\colon \mathrm{End}(\mathfrak{g}) \to \mathscr{A}(\mathfrak{g})$ is a $\mathfrak{g}$-module homomorphism.*

Proof. Let $X \in \mathfrak{g}$, $\Omega \in \mathfrak{g}^*_\omega$, $T \in \mathfrak{g}_t$ and $P \in I^s(\mathfrak{g})$. Then:

$$\begin{aligned}
\mathrm{L}^a_X(\tau_P(\Omega \underset{s}{\otimes} T)) &= \mathrm{L}^a_X(\Omega \wedge s(\mathrm{D}_T(P))) \\
&= \mathrm{L}^a_X \Omega \wedge s(\mathrm{D}_T(P)) + (-1)^{x\omega}\Omega \wedge \mathrm{L}^a_X(s(\mathrm{D}_T(P))) \\
&\overset{\text{Lemma 3.5}}{=} \check{\mathrm{ad}}_X \Omega \wedge s(D_T(P)) + (-1)^{x\omega}\Omega \wedge s(\mathrm{L}^s_X(\mathrm{D}_T(P))) \\
&= \check{\mathrm{ad}}_X \Omega \wedge s(\mathrm{D}_T(P)) + (-1)^{x\omega}\Omega \wedge (s([\mathrm{L}^s_X, \mathrm{D}_T](P)).
\end{aligned}$$

since $\mathrm{L}^s_X(P) = 0$. But the super derivations $[\mathrm{L}^s_X, \mathrm{D}_T]$ and $\mathrm{D}_{[X,T]}$ coincide: for $\phi \in \mathfrak{g}^*_\varphi$,

$$\begin{aligned}
[\mathrm{L}^s_X, \mathrm{D}_T](\phi) &= \mathrm{L}^s_X(\mathrm{D}_T(\phi)) - (-1)^{xt}\,\mathrm{D}_T(\mathrm{L}^s_X(\phi)) \\
&= 0 - (-1)^{xt+(x+\varphi)t}\check{\mathrm{ad}}_X(\phi)(T) \\
&= -(-1)^{\varphi t}\check{\mathrm{ad}}_X(\phi)(T) \\
&= (-1)^{\varphi(x+t)}\phi([X,T]) = \mathrm{D}_{[X,T]}(\phi).
\end{aligned}$$

Henceforth,

$$\begin{aligned}
\mathrm{L}^a_X(\tau_P(\Omega \underset{s}{\otimes} T)) &= \check{\mathrm{ad}}_X \Omega \wedge s(\mathrm{D}_T(P)) + (-1)^{x\omega}\Omega \wedge s(\mathrm{D}_{\mathrm{ad}X(T)}(P)) \\
&= \tau_P(\check{\mathrm{ad}}_X \Omega \underset{s}{\otimes} T + (-1)^{x\omega}\Omega \underset{s}{\otimes} \mathrm{ad}(X)(T)) \\
&= \tau_P((\check{\mathrm{ad}} \otimes \mathrm{ad})_X(\Omega \underset{s}{\otimes} T)).
\end{aligned}$$

□

As a direct consequence of (3.2) and Lemma 3.6, we obtain:

Corollary 3.7. *One has $t(I^s(\mathfrak{g})) \subset I^a(\mathfrak{g})$.*

Proof. Let $P \in I^s(\mathfrak{g})$. Then $t(P) = \tau_P(\mathrm{id})$. But $\mathrm{Id}_{\mathfrak{g}} = \sum_{i=1}^{p} \Omega_i \underset{s}{\otimes} X_i - \sum_{j=1}^{q} \phi_j \underset{s}{\otimes} Y_j$ is π-invariant, therefore using Lemma 3.6, $\mathrm{L}^a_X(t(P)) = 0$ for all $X \in \mathfrak{g}$. □

Denote $I^s_+(\mathfrak{g})$ the augmentation ideal of $I^s(\mathfrak{g})$.

Lemma 3.8. *For all $P \in I^s_+(\mathfrak{g})$, $s(P) = 0$.*

Proof. By Lemmas 3.2 and 3.4, $s(P) = \dfrac{1}{\deg_{\mathbb{Z}}(P)} d(t(P)) = 0$ because $t(P)$ is invariant, hence a cocycle. □

Finally applying Lemma 3.3, we can conclude

Lemma 3.9. *For all $P \in \mathbb{C} \oplus (I^s_+(\mathfrak{g}))^2$, $t(P) = 0$.*

Remark 3.10. *For similar results in the non graded case, see* [4] *or* [10].

Now, let $\mathfrak{a} = \mathfrak{a}_{\bar 0} \oplus \mathfrak{a}_{\bar 1}$ be a Lie super-subalgebra of $\mathfrak{g}$. Assume $\{X_1, \dots, X_p\}$ is a basis of $\mathfrak{g}_{\bar 0}$, $\{Y_1, \dots, Y_q\}$ a basis of $\mathfrak{g}_{\bar 1}$, $\{\Omega_1, \dots, \Omega_p\}$ and $\{\phi_1, \dots, \phi_q\}$ their respective dual basis such that $\{X_1, \dots, X_r\}$ is a basis of $\mathfrak{a}_{\bar 0}$, $\{Y_1, \dots, Y_s\}$ a basis of $\mathfrak{a}_{\bar 1}$ and $\{\Omega_1, \dots, \Omega_r\}$ and $\{\phi_1, \dots, \phi_s\}$ their respective dual basis. There are two transgression operators $t_{\mathfrak{g}}$ for $\mathfrak{g}$ and $t_{\mathfrak{a}}$ for $\mathfrak{a}$. The relation between these two operators is given by:

Proposition 3.11. *For all $n \geq 1$ and $P \in \mathrm{Sym}^n(\mathfrak{g})$,*

$$t_{\mathfrak{g}}(P)|_{\mathfrak{a}^{2n-1}} = t_{\mathfrak{a}}(P|_{\mathfrak{a}^n}).$$

Proof. Let $P \in \mathrm{Sym}^n(\mathfrak{g})$. We have:

$$t_{\mathfrak{g}}(P) = \sum_{k=1}^{p} \Omega_k \wedge s_{\mathfrak{g}}(\mathrm{D}_{X_k}(P)) - \sum_{\ell=1}^{q} \phi_\ell \wedge s_{\mathfrak{g}}(\mathrm{D}_{Y_\ell}(P)) \text{ and}$$

$$t_{\mathfrak{a}}(P|_{\mathfrak{a}^n}) = \sum_{k=1}^{r} \Omega_k \wedge s_{\mathfrak{g}}(\mathrm{D}_{X_k}(P|_{\mathfrak{a}^n})) - \sum_{\ell=1}^{s} \phi_\ell \wedge s_{\mathfrak{a}}(\mathrm{D}_{Y_\ell}(P|_{\mathfrak{a}^n}))$$

where $s_{\mathfrak{g}}$ (resp. $s_{\mathfrak{a}}$) is the homomorphism of algebras $\mathrm{Sym}(\mathfrak{g}^*) \to \mathrm{Ext}(\mathfrak{g}^*)$ (resp. $\mathrm{Sym}(\mathfrak{a}^*) \to \mathrm{Ext}(\mathfrak{a}^*)$) taking Ω_k to $d\Omega_k$ and ϕ_ℓ to $d\phi_\ell$ for $1 \leq k \leq p$ and $1 \leq \ell \leq q$ (resp. $1 \leq k \leq r$ and $1 \leq \ell \leq s$).

Let us examine a monomial $\Omega^I \phi^J = \Omega_1^{i_1} \cdot \ldots \cdot \Omega_p^{i_p} \cdot \phi_1^{j_1} \cdot \ldots \cdot \phi_q^{j_q} \in \mathrm{Sym}^n(\mathfrak{g})$: its restriction $(\Omega^I \phi^J)|_{\mathfrak{a}^n}$ is zero if one $i_{r+1}, \ldots, i_p, j_{s+1}, \ldots, j_q$ is not zero. Therefore $P|_{\mathfrak{a}^n}$ is a linear combination of monomials $\Omega^{I'} \phi^{J'}$ with $I' = (i_1, \ldots, i_r, 0, \ldots, 0) \in \mathbb{Z}^r$ and $J' = (j_1, \ldots, j_s, 0, \ldots, 0) \in \mathbb{Z}_2^s$.

On the other hand, it is immediate that:

$$\big(\Omega_k \wedge s(\mathrm{D}_{X_k}(P))\big)|_{\mathfrak{a}^{2n-1}} = 0, \ \forall\, k \geqslant r+1 \quad \text{ and } \quad \big(\phi_\ell \wedge s(\mathrm{D}_{Y_\ell}(P))\big)|_{\mathfrak{a}^{2n-1}} = 0, \ \forall\, \ell \geqslant s+1.$$

Hence

$$t_{\mathfrak{g}}(P)|_{\mathfrak{a}^{2n-1}} = \sum_{k=1}^{r} \big(\Omega_k \wedge s_{\mathfrak{g}}(\mathrm{D}_{X_k}(P))\big)|_{\mathfrak{a}^{2n-1}} - \sum_{\ell=1}^{s} \big(\phi_\ell \wedge s_{\mathfrak{g}}(\mathrm{D}_{Y_\ell}(P))\big)|_{\mathfrak{a}^{2n-1}}.$$

Now take $P = \Omega^I \phi^J$. Let $1 \leq k \leq r$ and $1 \leq \ell \leq s$. If $i_k \neq 0$ (i.e. if the term with index k appears in the sum), we have

$$\mathrm{D}_{X_k}(P) = i_k \Omega_1^{i_1} \cdot \ldots \cdot \Omega_k^{i_k - 1} \cdot \ldots \cdot \Omega_p^{i_p} \cdot \phi^J$$

then

$$\Omega_k \wedge s_{\mathfrak{g}}(\mathrm{D}_{X_k}(P)) = i_k \Omega_k \wedge (d\Omega_1)^{i_1} \wedge \ldots \wedge (d\Omega_k)^{i_k - 1} \wedge \ldots \wedge (d\Omega_p)^{i_p} \wedge (d\phi)^J$$

and if $j_\ell \neq 0$ (i.e. if the term with index ℓ appears in the sum), we have

$$\mathrm{D}_{Y_\ell}(P) = (-1)^{j_1 + \ldots + j_{\ell-1}} j_\ell \Omega^I \cdot \phi_1^{j_1} \cdot \ldots \cdot \phi_\ell^{j_\ell - 1} \cdot \ldots \cdot \phi_q^{j_q}$$

then

$$\phi_\ell \wedge s_{\mathfrak{g}}(\mathrm{D}_{Y_\ell}(P)) = (-1)^{j_1 + \ldots + j_{\ell-1}} j_\ell \phi_\ell \wedge (d\Omega)^I \wedge (d\phi_1)^{j_1} \wedge \ldots \wedge (d\phi_\ell)^{j_\ell - 1} \wedge \ldots \wedge (d\phi_q)^{j_q}.$$

Since $k \leqslant r$ and $\ell \leqslant s$, it follows:

$$\big(\Omega_k \wedge s_{\mathfrak{g}}(\mathrm{D}_{X_k}(P))\big)|_{\mathfrak{a}^{2n-1}} = 0$$

if one $i_{r+1}, \ldots, i_p, j_{s+1}, \ldots, j_q$ is not zero and the same works for $\big(\phi_\ell \wedge s_{\mathfrak{g}}(\mathrm{D}_{Y_\ell}(P))\big)|_{\mathfrak{a}^{2n+1}}$. (recall for instance that $d\Omega_k(X,Y) = -\Omega_k([X,Y]) = 0$ for all $X, Y \in \mathfrak{a}$ since $[X,Y] \in \mathfrak{a}$.

We can finally conclude that

$$t_{\mathfrak{g}}(P)|_{\mathfrak{a}^{2n-1}} = t_{\mathfrak{a}}(P|_{\mathfrak{a}^n}).$$

□

4. Standard super polynomials and super identities in $\mathfrak{gl}(p,q)$

In this section, $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with $\dim V_{\bar{0}} = p$, $\dim V_{\bar{1}} = q$, and $\mathfrak{g}$ is the Lie superalgebra $\mathfrak{g} = \mathrm{End}(V) \simeq \mathfrak{gl}(p,q)$. We identify $\mathrm{End}(V)$ and $V \otimes V^*$ by using:

$$(Z \otimes \Omega)(T) := Z\, \Omega(T), \ \forall\, Z, T \in V, \ \Omega \in V^*$$

Then define the super trace on $\mathfrak{g}$ as:

$$\mathrm{str}(Z \otimes \Omega) := (-1)^{\omega z} \Omega(Z), \ \forall\, Z \in V_z, \ \Omega \in V_\omega^*$$

Remark 4.1. *With this definition, the* 2*-form* $B(Z|T) := \mathrm{str}(ZT)$ *is supersymmetric and non degenerate on* $\mathfrak{g}$*. In the case* $p = 1$ *and* $q = 2n$*,* $B|_{\mathfrak{osp}(1,2n)}$ *is non degenerate as well.*

For $k \geqslant 1$, we define the *standard supersymmetric* polynomial $\mathscr{P}_k$ and the *standard skew supersymmetric* polynomial $\mathscr{A}_k$ as:

$$\begin{aligned}\mathscr{P}_k(X_1,\dots,X_k) &:= \sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma,\mathscr{X})X_{\sigma(1)}\dots X_{\sigma(k)},\\ \mathscr{A}_k(X_1,\dots,X_k) &:= \sum_{\sigma\in\mathfrak{S}_k}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})X_{\sigma(1)}\dots X_{\sigma(k)},\end{aligned}$$

where $X_1,\dots,X_k \in \mathfrak{g}$.

Remark that $\mathfrak{g}$ acts on $\mathscr{F}^p(\mathfrak{g},\mathfrak{g})$ by:

$$\begin{aligned}\pi_X(F)(X_1,\dots,X_p) &= \mathrm{ad}(X)(F(X_1,\dots,X_p))\\ &\quad -(-1)^{xf}\sum_{i=1}^{p}(-1)^{x(x_1+\dots+x_{i-1})}F(X_1,\dots,X_{i-1},[X,X_i],X_{i+1},\dots,X_p),\end{aligned}$$

where $X_1,\dots,X_p \in \mathfrak{g}$, $X \in \mathfrak{g}_x$ and $F \in \mathscr{F}^p_f(\mathfrak{g},\mathfrak{g})$.

Now define the product $\mu_k \in \mathscr{F}^k_{\bar{0}}(\mathfrak{g},\mathfrak{g})$ as $\mu_k(X_1,\dots,X_k) := X_1\dots X_k$, for $k\in\mathbb{N}$, $k\geq 1$. It is immediate that

$$\mathscr{P}_k = \mathrm{S}(\mu_k) \text{ and } \mathscr{A}_k = \mathrm{A}(\mu_k).$$

Moreover, it is easy to check that μ_k is $\mathfrak{g}$-invariant. Using this fact, one can prove that:

Proposition 4.2. *The polynomials $\mathscr{P}_k$ and $\mathscr{A}_k$ are $\mathfrak{g}$-invariant k-linear maps from $\mathfrak{g}^k$ to $\mathfrak{g}$.*

Proof. We have $\pi_X(\mu_k)(X_1,\dots,X_k) = 0$ for all $X,X_1,\dots,X_k \in \mathfrak{g}$. Then it is enough to show that A (resp. S) commute with π_X, that is, that π_X commutes with the skew supersymmetric (resp. supersymmetric) action. For $\sigma\in\mathfrak{S}_k$ et $F\in\mathscr{F}^k_f(\mathfrak{g},\mathfrak{g})$:

$$\begin{aligned}\pi_X(\sigma\underset{a}{\cdot}F)(X_1,\dots,X_k) &= \mathrm{ad}(X)((\sigma\underset{a}{\cdot}F)(X_1,\dots,X_k))\\ &\quad -(-1)^{xf}\sum_{i=1}^{k}(-1)^{x(x_1+\dots+x_{i-1})}(\sigma\underset{a}{\cdot}F)(X_1,\dots,[X,X_i],\dots,X_k)\\ &= \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\,\mathrm{ad}(X)(F(X_{\sigma(1)},\dots,X_{\sigma(k)}))\\ &\quad -(-1)^{xf}\sum_{i=1}^{k}(-1)^{x(x_1+\dots+x_{i-1})}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{Y}_i)F(Y^i_{\sigma(1)},\dots,Y^i_{\sigma(k)})\end{aligned}$$

with $Y^i_i = [X,X_i]$ (of degree $x+x_i$), $Y^i_\ell = X_\ell$ if $\ell\neq i$ and $\mathscr{Y}_i = (Y^i_1,\dots,Y^i_k)$. For $\sigma = (j\ j+1)\in\mathfrak{S}_k$, it results:

$$\begin{aligned}&\pi_X(\sigma\underset{a}{\cdot}F)(X_1,\dots,X_k)\\ =\ & -(-1)^{x_jx_{j+1}}\,\mathrm{ad}(X)(F(X_1,\dots,X_{j-1},X_{j+1},X_j,X_{j+2},\dots,X_k))\\ & +(-1)^{xf}\sum_{i=1}^{j-1}(-1)^{x(x_1+\dots+x_{i-1})}(-1)^{x_jx_{j+1}}F(X_1,\dots,[X,X_i],\dots,X_{j-1},X_{j+1},X_j,X_{j+2}\dots,X_k)\\ & +(-1)^{xf}(-1)^{x(x_1+\dots+x_{j-1})}(-1)^{(x+x_j)(x_{j+1})}F(X_1,\dots,X_{j-1},X_{j+1},[X,X_j],X_{j+2},\dots,X_k)\\ & +(-1)^{xf}(-1)^{x(x_1+\dots+x_j)}(-1)^{x_j(x+x_{j+1})}F(X_1,\dots,X_{j-1},[X,X_{j+1}],X_j,X_{j+2},\dots,X_k)\\ & +(-1)^{xf}\sum_{i=j+2}^{k}(-1)^{x(x_1+\dots+x_{i-1})}(-1)^{x_jx_{j+1}}F(X_1,\dots,X_{j-1},X_{j+1},X_j,X_{j+2},\dots,[X,X_i],\dots,X_k).\end{aligned}$$

On the other hand:

$$\begin{aligned}&(\sigma\underset{a}{\cdot}(\pi_X(F)))(X_1,\dots,X_k) = \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\pi_X(F)(X_{\sigma(1)},\dots,X_{\sigma(k)})\\ =\ & \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\,\mathrm{ad}(X)(F(X_{\sigma(1)},\dots,X_{\sigma(k)}))-\\ & (-1)^{xf}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\sum_{i=1}^{k}(-1)^{x(x_{\sigma(1)}+\dots+x_{\sigma(i-1)})}F(X_{\sigma(1)},\dots,X_{\sigma(i-1)},[X,X_{\sigma(i)}],X_{\sigma(i+1)},\dots,X_{\sigma(k)}).\end{aligned}$$

For $\sigma = (j\ j+1)$:

$$\begin{aligned}(\sigma \underset{a}{\cdot} (\pi_X(F)))\ (X_1,\dots,X_k) = \\ &-(-1)^{x_j x_{j+1}} \operatorname{ad}(X)(F(X_1,\dots,X_{j-1},X_{j+1},X_j,X_{j+2},\dots,X_k)) \\ &+(-1)^{xf}(-1)^{x_j x_{j+1}} \sum_{i=1}^{j-1} (-1)^{x(x_1+\dots+x_{i-1})} F(X_1,\dots,[X,X_i],\dots,X_{j-1},X_{j+1},X_j,X_{j+2},\dots,X_k) \\ &+(-1)^{xf}(-1)^{x_j x_{j+1}}(-1)^{x(x_1+\dots+x_{j-1})} F(X_1,\dots,X_{j-1},[X,X_{j+1}],X_j,X_{j+2},\dots,X_k) \\ &+(-1)^{xf}(-1)^{x_j x_{j+1}}(-1)^{x(x_1+\dots+x_{j-1}+x_{j+1})} F(X_1,\dots,X_{j-1},X_{j+1},[X,X_j],X_{j+2},\dots,X_k) \\ &+(-1)^{xf}(-1)^{x_j x_{j+1}} \sum_{i=j+2}^{k} (-1)^{x(x_1+\dots+x_{i-1})} F(X_1,\dots,X_{j-1},X_{j+1},X_j,X_{j+2},\dots,[X,X_i],\dots,X_k).\end{aligned}$$

Hence the equality works for all transpositions $(j\ j+1)$ and we can deduce

$$\sigma \underset{a}{\cdot} (\pi_X(F)) = \pi_X(\sigma \underset{a}{\cdot} F)$$

for all $\sigma \in \mathfrak{S}_k$. In the same manner, we can show:

$$\sigma \underset{s}{\cdot} (\pi_X(F)) = \pi_X(\sigma \underset{s}{\cdot} F)$$

for all $\sigma \in \mathfrak{S}_k$. Therefore

$$\mathrm{S}(\pi_X(F)) = \pi_X(\mathrm{S}(F)) \quad \text{and} \quad \mathrm{A}(\pi_X(F)) = \pi_X(\mathrm{A}(F)).$$

□

It is clear that $\mathscr{P}_k$ is supersymmetric, that is,

$$\mathscr{P}_k(X_{\sigma(1)},\dots,X_{\sigma(k)}) = \varepsilon(\sigma,\mathscr{X})\mathscr{P}_k(X_1,\dots,X_k).$$

Similarly, the polynomial $\mathscr{A}_k$ is skew supersymmetric, meaning:

$$\mathscr{A}_k(X_{\sigma(1)},\dots,X_{\sigma(k)}) = \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\mathscr{A}_k(X_1,\dots,X_k).$$

Moreover they verify the recursive relations below:

Proposition 4.3.

(1) $\mathscr{P}_{k+1}(X_1,\dots,X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{x_j(x_1+\dots+x_{j-1})} X_j\ \mathscr{P}_k(X_1,\dots,\widehat{X_j},\dots,X_{k+1})$,

(2) $\mathscr{A}_{k+1}(X_1,\dots,X_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1}(-1)^{x_j(x_1+\dots+x_{j-1})} X_j \mathscr{A}_k(X_1,\dots,\widehat{X_j},\dots,X_{k+1})$.

Proof. We will show the second equation *(2)*. Let $\mathscr{X} = (X_1,\dots,X_{k+1}) \in \mathfrak{g}^{k+1}$, $\mathscr{X}_i = (X_1,\dots,X_{i-1},X_{i+1},\dots,X_{k+1}) \in \mathfrak{g}^k$ and $S_{k+1}^i = \{\sigma \in \mathfrak{S}_{k+1} \mid \sigma(1) = i\}$, $i = 1,\dots,k+1$. We have then the partition $\mathfrak{S}_{k+1} = \bigsqcup_{i=1}^{k+1} S_{k+1}^i$. On the other hand, there is a one-to-one correspondence between S_{k+1}^i and the symmetric group $\mathfrak{S}_k^i$ of all permutations of $\{1,\dots,i-1,i+1,\dots,k+1\}$: to $\sigma \in S_{k+1}^i$ we associate $\widetilde{\sigma} \in \mathfrak{S}_k^i$ with

$$\widetilde{\sigma} = \begin{pmatrix} 1 & 2 & \dots & i-1 & i+1 & \dots & k+1 \\ \sigma(2) & \sigma(3) & \dots & \sigma(i) & \sigma(i+1) & \dots & \sigma(k+1) \end{pmatrix}.$$

Fix $1 \le i \le k+1$ and $\sigma \in S_{k+1}^i$. We want to write the sign and the super sign of $\widetilde{\sigma}$ using σ. The inversions of $\widetilde{\sigma}$ are:

- $1 \leqslant r < s \leqslant i-1$ with $\sigma(r+1) > \sigma(s+1)$ i.e. $2 \leqslant r < s \leqslant i$ with $\sigma(r) > \sigma(s)$;
- $1 \leqslant r < i < s \leqslant k+1$ with $\sigma(r+1) > \sigma(s)$ i.e. $2 \leqslant r \leqslant i < s \leqslant k+1$ with $\sigma(r) > \sigma(s)$;
- $i+1 \leqslant r < s \leqslant k+1$ with $\sigma(r) > \sigma(s)$.

We obtain all inversions of σ except $(1\ s)$ with $\sigma(s) \in \{1,2,\dots,i-1\}$ ($\sigma(s) < i = \sigma(1)$). We conclude:

$$\varepsilon(\sigma) = (-1)^{i-1}\varepsilon(\widetilde{\sigma}) \text{ and } \varepsilon(\sigma,\mathscr{X}) = (-1)^{x_i(x_1+\dots+x_{i-1})}\varepsilon(\widetilde{\sigma},\mathscr{X}_i).$$

Hence:

$$\begin{aligned}\mathscr{A}_{k+1}(\mathscr{X}) &= \sum_{i=1}^{k+1}(-1)^{i+1}(-1)^{x_i(x_1+\ldots+x_{i-1})}X_i \sum_{\widetilde{\sigma}\in\mathfrak{S}_k^i}\varepsilon(\widetilde{\sigma})\varepsilon(\widetilde{\sigma},\mathscr{X}_i)X_{\widetilde{\sigma}(1)}\ldots X_{\widetilde{\sigma}(i-1)}X_{\widetilde{\sigma}(i+1)}\ldots X_{\widetilde{\sigma}(k)} \\ &= \sum_{i=1}^{k+1}(-1)^{i+1}(-1)^{x_i(x_1+\ldots+x_{i-1})}X_i\ \mathscr{A}_k(X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_{k+1}).\end{aligned}$$

The first equation can be shown in the same manner by removing all terms concerning the sign. □

From $\mathscr{P}_k$ and $\mathscr{A}_k$, we can construct $P_k\in I^s(\mathfrak{g})$ and $\Lambda_k\in I^a(\mathfrak{g})$:

$$\begin{aligned}P_k(X_1,\ldots,X_k) &:= \operatorname{str}(\mathscr{P}_k(X_1,\ldots,X_k)), \\ \Lambda_k(X_1,\ldots,X_k) &:= \operatorname{str}(\mathscr{A}_k(X_1,\ldots,X_k))\end{aligned}$$

Proposition 4.4. *The multilinear forms P_k and Λ_k are $\mathfrak{g}$-invariant.*

Proof. Let π'_X denote the action of $X\in\mathfrak{g}_x$ on $\mathscr{F}(\mathfrak{g})$. Since str is ad-invariant, we have:

$$\begin{aligned}&\pi'_X(\Lambda_k)(X_1,\ldots,X_k) \\ =\ & -\operatorname{str}\left(\sum_{i=1}^{k}(-1)^{x(x_1+\ldots+x_{i-1})}\mathscr{A}_k(X_1,\ldots,X_{i-1},[X,X_i],X_{i+1},\ldots,X_k)\right) \\ =\ & \operatorname{str}\left(\operatorname{ad}(X)(\mathscr{A}_k(X_1,\ldots,X_k))-\sum_{i=1}^{k}(-1)^{x(x_1+\ldots+x_{i-1})}\mathscr{A}_k(X_1,\ldots,X_{i-1},[X,X_i],X_{i+1},\ldots,X_k)\right) \\ =\ & \operatorname{str}(\pi_X(\mathscr{A}_k)(X_1,\ldots,X_k))=0.\end{aligned}$$

Same reasoning works for P_k. □

Proposition 4.5. *For $X_1,\ldots,X_{2k+1}\in\mathfrak{g}$, one has:*

$$\begin{aligned}&P_{2k+1}(X_1,\ldots,X_{2k+1})=(2k+1)B(\mathscr{P}_{2k}(X_1,\ldots,X_{2k})|X_{2k+1}), \\ &\Lambda_{2k}(X_1,\ldots,X_{2k})=0, \\ &\Lambda_{2k+1}(X_1,\ldots,X_{2k+1})=(2k+1)B(\mathscr{A}_{2k}(X_1,\ldots,X_{2k})|X_{2k+1}).\end{aligned}\tag{4.3}$$

Proof. We will use Remark 4.1, that is, the super trace is supersymmetric: $\operatorname{str}(XY)=(-1)^{xy}\operatorname{str}(YX)$, for $X\in\mathfrak{g}_x$ and $Y\in\mathfrak{g}_y$.

As before, let $\mathscr{X}=(X_1,\ldots,X_n)\in\mathfrak{g}^n$ and $S_n^i=\{\sigma\in\mathfrak{S}_n\mid\sigma(i)=n\}$, for $1\le i\le n$. Since $\mathfrak{S}_n=\bigsqcup_{i=1}^{n}S_n^i$, it follows:

$$\begin{aligned}&\Lambda_n(X_1,\ldots,X_n)=\sum_{\sigma\in\mathfrak{S}_n}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\operatorname{str}(X_{\sigma(1)}\ldots X_{\sigma(n)}) \\ =\ & \sum_{i=1}^{n}\sum_{\sigma\in S_n^i}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\operatorname{str}\big((X_{\sigma(1)}\ldots X_{\sigma(i-1)}X_n)(X_{\sigma(i+1)}\ldots X_{\sigma(n)})\big) \\ =\ & \sum_{i=1}^{n}\sum_{\sigma\in S_n^i}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})(-1)^{(x_{\sigma(1)}+\ldots+x_{\sigma(i)})(x_{\sigma(i+1)}+\ldots+x_{\sigma(n)})}\operatorname{str}(X_{\sigma(i+1)}\ldots X_{\sigma(n)}X_{\sigma(1)}\ldots X_{\sigma(i-1)}X_n)\end{aligned}$$

But there is an one-one correspondence between S_n^i and the symmetric group $\mathfrak{S}_{n-1}$ given by $\sigma\in S_n^i\mapsto\widetilde{\sigma}=(\sigma\pi^i)_{\{1,\ldots,n-1\}}\in\mathfrak{S}_{n-1}$ where $\pi=(1\ 2\ \ldots\ n)\in\mathfrak{S}_n$. Therefore $\varepsilon(\widetilde{\sigma})=(-1)^{i(n-1)}\varepsilon(\sigma)$ for $\sigma\in S_n^i$ whereas $\varepsilon(\pi)=(-1)^{n-1}$.

Notice that

$$\pi^i=\begin{pmatrix}1 & 2 & \ldots & n-i & n-i+1 & \ldots & n \\ i+1 & i+2 & \ldots & n & 1 & \ldots & i\end{pmatrix}$$

then $\varepsilon(\pi^i,\mathscr{Y}) = (-1)^{(y_{i+1}+\ldots+y_n)(y_1+\ldots+y_i)}$ for $\mathscr{Y} = (Y_1,\ldots,Y_n) \in \mathfrak{g}^n$. As a consequence:

$$\begin{aligned}\Lambda_n(X_1,\ldots,X_n) &= \sum_{i=1}^{n}\sum_{\sigma\in S_n^i} \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})\varepsilon(\pi^i,\sigma^{-1}\cdot\mathscr{X})\,\mathrm{str}(X_{\sigma\pi^i(1)}\ldots X_{\sigma\pi^i(n-1)}X_n)\\ &= \sum_{i=1}^{n}(-1)^{i(n-1)}\,\mathrm{str}\left(\sum_{\widetilde{\sigma}\in\mathfrak{S}_{n-1}} \varepsilon(\widetilde{\sigma})\varepsilon(\widetilde{\sigma},\mathscr{X})X_{\widetilde{\sigma}(1)}\ldots X_{\widetilde{\sigma}(n-1)}\;X_n\right)\\ &= \mathrm{str}(\mathscr{A}_{n-1}(X_1,\ldots,X_{n-1})X_n)\sum_{i=1}^{n}(-1)^{i(n-1)}.\end{aligned}$$

To obtain the second formula, set $n = 2k$:

$$\sum_{i=1}^{2k}(-1)^{i(2k-1)} = \sum_{i=1}^{2k}(-1)^i = 0$$

and for the third identity, $n = 2k+1$:

$$\sum_{i=1}^{2k+1}(-1)^{i(2k)} = 2k+1.$$

The first formula can be obtained by the same computation as above by removing the sign. ☐

Proposition 4.6. *For $X_1,\ldots,X_{2k} \in g$, one has:*

$$\sum_{\sigma\in\mathfrak{S}_{2k}} \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})[X_{\sigma(1)},X_{\sigma(2)}]\ldots[X_{\sigma(2k-1)},X_{\sigma(2k)}] = 2^k\mathscr{A}_{2k}(X_1,\ldots,X_{2k}) \tag{4.4}$$

Proof. For $1 \le i \le k$, set $\tau_i := (2i-1\;\; 2i) \in \mathfrak{S}_{2k}$ and for $\alpha = (\alpha_1,\ldots,\alpha_k) \in \{0,1\}^k$, $\sigma_\alpha := \prod_{i=1}^{k}\tau_i^{\alpha_i}$. We have then for all $\mathscr{Y} = (Y_1,\ldots,Y_{2k}) \in \mathfrak{g}^{2k}$:

$$\varepsilon(\sigma_\alpha) = (-1)^{\alpha_1+\ldots+\alpha_k} \text{ and } \varepsilon(\sigma_\alpha,\mathscr{Y}) = (-1)^{\alpha_1 y_1 y_2+\ldots+\alpha_k y_{2k-1}y_{2k}}$$

Since $[Y_1,Y_2] = Y_1Y_2 - (-1)^{y_1y_2}Y_2Y_1$, it follows:

$$\begin{aligned}[][Y_1,Y_2]\ldots[Y_{2k-1},Y_{2k}] &= \sum_{\alpha\in\{0,1\}^k}\varepsilon(\sigma_\alpha)\varepsilon(\sigma_\alpha,\mathscr{Y})Y_{\sigma_\alpha(1)}\ldots Y_{\sigma_\alpha(2k)}\\ &= \sum_{\alpha\in\{0,1\}^k}\sigma_\alpha \underset{a}{\cdot}\mu_{2k}(Y_1,\ldots,Y_{2k}).\end{aligned}$$

Let $\sigma\in\mathfrak{S}_{2k}$. If $\mathscr{Y} = \sigma^{-1}\cdot\mathscr{X}$, then:

$$[X_{\sigma(1)},X_{\sigma(2)}]\ldots[X_{\sigma(2k-1)},X_{\sigma(2k)}] = \sum_{\alpha\in\{0,1\}^k}\sigma_\alpha \underset{a}{\cdot}\mu_{2k}(\sigma^{-1}\cdot\mathscr{X})$$

and:

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_{2k}}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})[X_{\sigma(1)},X_{\sigma(2)}]\ldots[X_{\sigma(2k-1)},X_{\sigma(2k)}] &= \sum_{\alpha\in\{0,1\}^k}\sum_{\sigma\in\mathfrak{S}_{2k}}\sigma\underset{a}{\cdot}(\sigma_\alpha\underset{a}{\cdot}\mu_{2k})(\mathscr{X})\\ &= \sum_{\alpha\in\{0,1\}^k}\sum_{\sigma\in\mathfrak{S}_{2k}}(\sigma\sigma_\alpha)\underset{a}{\cdot}\mu_{2k}(\mathscr{X})\\ &= \sum_{\alpha\in\{0,1\}^k}\mathscr{A}_{2k}(X_1,\ldots,X_{2k})\\ &= 2^k\mathscr{A}_{2k}(X_1,\ldots,X_{2k}).\end{aligned}$$

☐

Proposition 4.7. *For all $\ell \in \mathbb{N}$, $0 \leq \ell \leq k$, one has:*

$$\sum_{\sigma\in\mathfrak{S}_{2k+1}} \varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X}) \quad [X_{\sigma(1)},X_{\sigma(2)}]\dots[X_{\sigma(2\ell-1)},X_{\sigma(2\ell)}]X_{\sigma(2\ell+1)}\times$$
$$\times[X_{\sigma(2\ell+2)},X_{\sigma(2\ell+3)}]\dots[X_{\sigma(2k)},X_{\sigma(2k+1)}] = 2^k\mathscr{A}_{2k+1}(X_1,\dots,X_{2k+1}) \tag{4.5}$$

Proof. We proceed as in Proposition 4.6: let $\tau_i = (2i-1 \;\; 2i) \in \mathfrak{S}_{2k+1}$ for $1 \leq i \leq \ell$, $\tau_i = (2i \;\; 2i+1) \in \mathfrak{S}_{2k+1}$ for $\ell+1 \leq i \leq k$ and σ_α defined as previously. In particular, $\sigma_\alpha(2\ell+1) = 2\ell+1$. We have then:

$$\varepsilon(\sigma_\alpha,\mathscr{Y}) = (-1)^{\alpha_1 y_1 y_2+\dots+\alpha_\ell y_{2\ell-1}y_{2\ell}+\alpha_{\ell+1}y_{2\ell+2}y_{2\ell+3}+\dots+\alpha_k y_{2k}y_{2k+1}}.$$

Hence:

$$[Y_1,Y_2]\dots[Y_{2\ell-1},Y_{2\ell}]Y_{2\ell+1}[Y_{2\ell+2},Y_{2\ell+3}]\dots[Y_{2k},Y_{2k+1}] = \sum_{\alpha\in\{0,1\}^k} \sigma_\alpha \underset{a}{\cdot} \mu_{2k+1}(\mathscr{Y}).$$

We conclude as in the previous Proposition. □

Remark 4.8. *The identities (4.3), (4.4) and (4.5) are super versions of classical identities in the non graded case. Other super identities can be settled, but they will not be needed in this work.*

Let us examine what happens when we apply the transgression on the invariant P_k defined by the super trace. This is a super version of Dynkin's formula.

Theorem 4.9. *One has $t(P_k) = (-1)^{k-1}k\Lambda_{2k-1}$.*

Proof. The main argument here will be Lemma 3.3. Let M_{ij} be the coordinate forms. Then

$$M_{ii}(X_1\dots X_k) = \sum_{R=(r_1,\dots,r_{k-1})} (-1)^{\Delta(m_{iR},m_{iR})} M_{ir_1} \underset{s}{\otimes} \dots \underset{s}{\otimes} M_{r_{k-1}i}(X_1,\dots,X_k)$$

where $m_{iR} := \begin{pmatrix} m_{ir_1} \\ \vdots \\ m_{r_{k-1}i} \end{pmatrix}$.

Supersymmetrizing, we obtain:

$$P_k = \sum_{\substack{R \\ i\,\in\,\{1,\dots,p\}}} (-1)^{\Delta(m_{iR},m_{iR})} M_{ir_1}\cdot M_{r_1r_2}\cdot\dots\cdot M_{r_{k-1}i} - \sum_{\substack{R \\ j\,\in\,\{p+1,\dots,p+q\}}} (-1)^{\Delta(m_{jR},m_{jR})} M_{jr_1}\cdot M_{r_1r_2}\cdot\dots\cdot M_{r_{k-1}j}$$

(notice that the products above are calculated in $\mathscr{P}(\mathfrak{g})$).

From $t(M_{rs}) = M_{rs}$, $\forall\, r,s$ and Lemma 3.3, we have:

$$t(M_{ir_1}\cdot\dots\cdot M_{r_{k-1}i}) = \sum_{\ell=1}^{k} dM_{ir_1}\wedge dM_{r_1r_2}\wedge\dots\wedge M_{r_{\ell-1}r_\ell}\wedge\dots\wedge dM_{r_{k-1}i}$$

(if $\ell = k$ then $r_k = i$ in the sum).

Therefore:

$$t(M_{ir_1}\cdot\dots\cdot M_{r_{k-1}i})(X_1,\dots,X_{2k-1}) = (-1)^{\Delta(m_{iR},m_{iR})} \frac{(-1)^{k-1}}{2^{k-1}}\sum_{\sigma,\ell}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})M_{ir_1}([X_{\sigma(1)},X_{\sigma(2)}])\dots$$
$$M_{r_{\ell-1}r_\ell}(X_{\sigma(2\ell-1)})\dots M_{r_{k-1}i}([X_{\sigma(2k-2)},X_{\sigma(2k-1)}])$$

At the end, we have:

$$\begin{aligned}\sum_R(-1)^{\Delta(m_{iR},m_{iR})}t(M_{ir_1}\cdot\dots\cdot M_{r_{k-1}i})(X_1,\dots,X_{2k-1}) &= \frac{(-1)^{k-1}}{2^{k-1}}\sum_{\sigma,R,\ell}\varepsilon(\sigma)\varepsilon(\sigma,\mathscr{X})M_{ir_1}([X_{\sigma(1)},X_{\sigma(2)}])\dots \\ &\quad M_{r_{\ell-1}r_\ell}(X_{\sigma(2\ell-1)})\dots M_{r_{k-1}i}([X_{\sigma(2k-2)},X_{\sigma(2k-1)}]) \\ &= (-1)^{k-1}\sum_\ell M_{ii}(\mathscr{A}_{2k-1}(X_1,\dots,X_{2k-1})) \qquad \text{(by 4.5)} \\ &= (-1)^{k-1}kM_{ii}(\mathscr{A}_{2k-1}(X_1,\dots,X_{2k-1})).\end{aligned}$$

□

5. The Amitsur Levitzki theorem for $\mathfrak{osp}(1,2n)$

Henceforth we will assume that $\mathfrak{g} = \mathfrak{osp}(1,2n)$ and $\widetilde{\mathfrak{g}} = \mathfrak{gl}(1,2n)$. We will now prove a (super) version of the Amitsur-Levitzki theorem for $\mathfrak{g}$. In other words, we will show:

Theorem 5.1. *For all $X_1,\dots,X_{4n+2} \in \mathfrak{g}$, one has $\mathscr{A}_{4n+2}(X_1,\dots,X_{4n+2}) = 0$.*

Notice that this identity is valid if $X_1,\dots,X_{4n+2} \in \mathfrak{g}_{\bar 0}$ by the classical Amitsur-Levitzki theorem. Furthermore, if $X_1 = \cdots = X_{4n+2} = X \in \mathfrak{g}_{\bar 1}$ then by Proposition 2.1, the identity holds as well.

The theorem will be a consequence of Theorem 4.9 and two lemmas:

Lemma 5.2. *One has:*

(1) *For all $X_1,\dots,X_{2p+1} \in \mathfrak{g}$, $\mathscr{P}_{2p+1}(X_1,\dots,X_{2p+1}) \in \mathfrak{g}$.*
(2) *For all $X_1,\dots,X_{4k+1} \in \mathfrak{g}$, $\mathscr{A}_{4k+1}(X_1,\dots,X_{4k+1}) \in \mathfrak{g}$.*
(3) *For all $X_1,\dots,X_{4k+2} \in \mathfrak{g}$, $\mathscr{A}_{4k+2}(X_1,\dots,X_{4k+2}) \in \mathfrak{g}$.*

Recall that if V is $2n+1$-dimensional vector space, the Lie superalgebra $\mathfrak{osp}(1,2n)$ is the subalgebra of $\mathfrak{gl}(V)$ that leaves invariant a super symmetric non degenerate bilinear form F:

$$\mathfrak{osp}(1,2n) = \mathfrak{osp}(1,2n)_{\bar 0} \oplus \mathfrak{osp}(1,2n)_{\bar 1}$$

with

$$\mathfrak{osp}(1,2n)_t = \{T \in \mathfrak{gl}(1,2n)_t \mid F(T(X),Y) + (-1)^{tx} F(X,T(Y)) = 0\ \forall\, X \in V_x,\ Y \in V\}$$

for $t \in \mathbb{Z}_2$.

Proof. We will use the notation above. Let $Y \in V_y$, $Z \in V_z$ and $Y_1,\dots,Y_m \in \mathfrak{osp}(1,2n)$. Set $\mathscr{Y} = (Y_1,\dots,Y_m)$. Since $Y_1,\dots,Y_m \in \mathfrak{osp}(1,2n)$:

$$\begin{aligned} F(Y_1Y_2\dots Y_mY,Z) &= -(-1)^{y_1(y_2+\dots+y_m+y)} F(Y_2\dots Y_mY, Y_1Z) \\ &= \dots \\ &= (-1)^m(-1)^{y_1(y_2+\dots+y_m+y)}(-1)^{y_2(y_3+\dots+y_m+y)}\dots(-1)^{y_my}F(Y,Y_m\dots Y_1Z). \end{aligned}$$

Let $\sigma = \begin{pmatrix} 1 & 2 & \dots & m-1 & m \\ m & m-1 & \dots & 2 & 1 \end{pmatrix} \in \mathfrak{S}_m$. Then $\sigma\cdot\mu_m(Y_1,\dots,Y_m) = \mu_m(Y_m,\dots,Y_1) = Y_m\dots Y_1$.

- If m is even ($m = 2p$):

  $\sigma = (1\ \ 2p)(2\ \ 2p-1)\dots(p\ \ p+1)$ then $\varepsilon(\sigma) = (-1)^p$ and the inversions of σ are all $(i\ \ j)$ with $i<j$, $i,j = 1,\dots,2p$. Therefore:

  $$\varepsilon(\sigma,\mathscr{Y}) = (-1)^{y_1(y_2+\cdots+y_{2p})+y_2(y_3+\cdots+y_{2p})+\cdots+y_{2p-1}y_{2p}}.$$

  Hence:

  $$F(\mu_{2p}(Y_1,\dots,Y_{2p})Y,Z) = (-1)^p(-1)^{y(y_1+\cdots+y_{2p})}F(Y,(\sigma \underset{a}{\cdot} \mu_{2p})(Y_1,\dots,Y_{2p})).$$

  In particular, for $\mathscr{Y} = \pi^{-1}\cdot(X_1,\dots,X_{2p})$ multiplying the two members of the expression above by $\varepsilon(\pi)\varepsilon(\pi,\mathscr{X})$, we obtain:

  $$\begin{aligned} F((\pi \underset{a}{\cdot} \mu_{2p})(X_1,\dots,X_{2p})Y,Z) &= (-1)^p(-1)^{y(x_1+\dots+x_{2p})}F(Y,(\pi \underset{a}{\cdot}(\sigma \underset{a}{\cdot} \mu_{2p}))(X_1,\dots,X_{2p})) \\ &= (-1)^p(-1)^{y(x_1+\dots+x_{2p})}F(Y,((\pi\sigma) \underset{a}{\cdot} \mu_{2p})(X_1,\dots,X_{2p})). \end{aligned}$$

  $\diamond$ If p is odd ($p = 2k+1$, $m = 4k+2$), adding over $\mathfrak{S}_{4k+2}$, it results:

  $$\begin{aligned} F(\mathscr{A}_{4k+2}(X_1,\quad\dots\quad,X_{4k+2})Y,Z) &= \sum_{\pi\in\mathfrak{S}_{4k+2}} F((\pi \underset{a}{\cdot} \mu_{4k+2})(X_1,\dots,X_{4k+2})Y,Z) \\ &= -(-1)^{y(x_1+\dots+x_{4k+2})}\sum_{\pi\in\mathfrak{S}_{4k+2}} F(Y,((\pi\sigma) \underset{a}{\cdot} \mu_{4k+2})(X_1,\dots,X_{4k+2})) \\ &= -(-1)^{y(x_1+\dots+x_{4k+2})}F(Y,\mathscr{A}_{4k+2}(X_1,\dots,X_{4k+2})Z). \end{aligned}$$

  Then $\mathscr{A}_{4k+2}(X_1,\dots,X_{4k+2}) \in \mathfrak{osp}(1,2n)$.

We can also deduce:

$$F(\mathscr{P}_{4k+2}(X_1,\dots,X_{4k+2})Y,Z) = (-1)^{y(x_1+\dots+x_{4k+2})}F(Y,\mathscr{P}_{4k+2}(X_1,\dots,X_{4k+2})Z)$$

since $\varepsilon(\sigma) = (-1)^p = -1$ does not contribute to the expression.

$\diamond$ If p is even ($p = 2k$), we have:

$$F(\mathscr{A}_{4k}(X_1,\dots,X_{4k})Y,Z) = (-1)^{y(x_1+\dots+x_{4k})}F(Y,\mathscr{A}_{4k}(X_1,\dots,X_{4k})Z)$$

and the same identity follows by replacing $\mathscr{A}_{4k}$ by $\mathscr{P}_{4k}$.

- If m is odd ($m = 2p+1$):

$\sigma = (1\ \ 2p+1)(2\ \ 2p)\dots(p\ \ p+2)$ then $\varepsilon(\sigma) = (-1)^p$. The inversions of σ are the same as in the preceding case and that gives us again:

$$\varepsilon(\sigma,\mathscr{Y}) = (-1)^{y_1(y_2+\dots+y_{2p+1})+y_2(y_3+\dots+y_{2p+1})+\dots+y_{2p}y_{2p+1}}.$$

The sign $\varepsilon(\sigma)$ does not contribute to the computation of $\mathscr{P}_{2p+1}$, and it follows immediately that:

$$\mathscr{P}_{2p+1}(X_1,\dots,X_{2p+1}) \in \mathfrak{osp}(1,2n)$$

since $(-1)^m = -1$.

$\diamond$ If p is even ($p = 2k$, $m = 4k+1$): $(-1)^p = 1$ but $(-1)^m = -1$ therefore:

$$F(\pi \underset{a}{\cdot} \mu_{4k+1}(X_1,\dots,X_{4k+1})Y,Z) = -(-1)^{y(x_1+\dots+x_{4k+1})}F(Y,(\pi\sigma)\underset{a}{\cdot}\mu_{4k+1}(X_1,\dots,X_{4k+1})).$$

Henceforth $\mathscr{A}_{4k+1}(X_1,\dots,X_{4k+1}) \in \mathfrak{osp}(1,2n)$.

$\diamond$ If p is odd ($p = 2k+1$, $m = 4k+3$): $(-1)^m(-1)^p = 1$ then $\mathscr{A}_{4k+3}(X_1,\dots,X_{4k+3})$ is super symmetric with respect to F.

□

As a consequence, P_{2p+1}, Λ_{4k+1} and Λ_{4k+2} vanish as multilinear mappings on $\mathfrak{g}$.

Recall from Subsection 2.3 that the restriction $\mathrm{Res}\colon I^s(\mathfrak{g}) \to J$ is an algebra isomorphism where $J := \mathrm{Sym}(\mathfrak{h}^*)^W$. Moreover, we can write $J = \mathbb{C}[t_1,\dots,t_n]$ with $t_k = \sum_{i=1}^n \alpha_i^{2k}$ for $k \in \mathbb{N}$, $k \geq 1$ and $\{\alpha_1,\dots,\alpha_n\}$ simple roots of $\mathfrak{g}$.

Lemma 5.3. *One has* $\mathrm{Res}(P_{2k}) = 2(2k)!\, t_k$.

Proof. Recall that if $\{E_{i,j}, 1 \leqslant i,j \leqslant 2n+1\}$ is the canonical basis of $\mathfrak{gl}(2n+1,\mathbb{C})$, then $\{H_i = E_{i,i} - E_{n+i,n+i} \mid i = 1,\dots,n\}$ is a basis for a Cartan subalgebra $\mathfrak{h}$. We remark that $H_iH_j = 0$ if $i \neq j$. Then P_k is zero on the k-tuples $(H_{i_1},\dots,H_{i_k})$ if $\sharp(\{i_1,\dots,i_k\}) > 1$. We have:

$$P_k(H_i,\dots,H_i) = k!\mathrm{tr}(H_i\dots H_i).$$

The product of k matrices H_i is equal to $E_{i,i} + (-1)^k E_{n+i,n+i}$ therefore $P_k(H_i,\dots,H_i) = k!(1+(-1)^k)$. We can deduce that:

$$\mathrm{Res}(P_k) = k!\sum_{i=1}^n (1+(-1)^k)\alpha_i^k$$

i.e. $\mathrm{Res}(P_{2k+1}) = 0$ and $\mathrm{Res}(P_{2k}) = 2(2k)!\, t_k$. □

Corollary 5.4. *One has* $I^s(\mathfrak{g}) = \mathbb{C}[P_2,P_4,\dots,P_{2n}]$ *and* $P_{2n+2} \in (I^s_+(\mathfrak{g}))^2$.

We will next terminate the proof of Theorem 5.1.

Proof. (of Theorem 5.1)

Let $t_{\mathfrak{g}}$ be the transgression defined on $\mathfrak{g}$ and $t_{\widetilde{\mathfrak{g}}}$ be transgression defined on $\widetilde{\mathfrak{g}}$. Since $\mathfrak{g}$ is a subalgebra of $\widetilde{\mathfrak{g}}$, if P is a p-form in $\mathscr{P}(\widetilde{\mathfrak{g}})$, one has $t_{\widetilde{\mathfrak{g}}}(P)|_{\mathfrak{g}^p} = t_{\mathfrak{g}}(P|_{\mathfrak{g}^p})$ (by Proposition 3.11). In the sequel, we use t for both transgressions $t_{\mathfrak{g}}$ and $t_{\widetilde{\mathfrak{g}}}$, and we consider multilinear mappings restricted to $\mathfrak{g}$.

Now, since $P_{2n+2} \in (I^s_+(\mathfrak{g}))^2$ (by Corollary 5.4), we have $t(P_{2n+2}) = 0$ from Lemma 3.9. Using Theorem 4.9, we deduce $t(P_{2n+2}) = -(2n+2)\Lambda_{4n+3}$, hence $\Lambda_{4n+3} = 0$. From Proposition 4.5, for all $X_1,\dots,X_{4n+3} \in \mathfrak{g}$,

$$\Lambda_{4n+3}(X_1,\dots,X_{4n+3}) = (4n+3)B(\mathscr{A}_{4n+2}(X_1,\dots,X_{4n+2})|X_{4n+3}).$$

But $\mathscr{A}_{4n+2}(X_1,\dots,X_{4n+2}) \in \mathfrak{g}$ by Lemma 5.2 *(3)*, hence from Remark 4.1:

$$\mathscr{A}_{4n+2}(X_1,\dots,X_{4n+2}) = 0, \text{ for all } X_1,\dots,X_{4n+2} \in \mathfrak{g}.$$

□

Proposition 5.5. *The polynomial $\mathscr{A}_{4n}$ is not always zero on $\mathfrak{g}$.*

Proof. Assume $\mathscr{A}_{4n}$ is identically zero. Using the realization of $\mathfrak{g}$ in the Weyl algebra A_n (see 2.1), we are able to compute the twisted action of $\mathscr{A}_{4n}(X_1,\dots,X_{4n-1},X)$ of $\mathfrak{g}_{\bar{1}}$, with $X_1,\dots,X_{4n-1} \in \mathfrak{g}_{\bar{0}}$ and $X \in \mathfrak{g}_{\bar{1}}$:

$$\mathscr{A}_{4n}(X_1,\dots,X_{4n-1},X)(1) = 2\ S_{4n-1}(X_1,\dots,X_{4n-1})(X).$$

Indeed, we have:

$$\mathrm{ad}'(X)(1) = 2X \qquad \text{and} \qquad \mathrm{ad}'(X_i)(1) = [X_i,1]_{\mathscr{L}} = 0,\ \forall\ i = 1,\dots,4n-1,$$

where S_n denotes the classical standard polynomial.

As a consequence, since $\mathscr{A}_{4n}$ is identically zero, we deduce $S_{4n-1}(X_1,\dots,X_{4n-1})$ is always zero as an operator on elements of degree $\bar{1}$. But it is also zero on elements of degree $\bar{0}$ (all multiples of 1), therefore:

$$S_{4n-1}(X_1,\dots,X_{4n-1}) = 0$$

for all $X_1,\dots,X_{4n-1} \in \mathfrak{g}_{\bar{0}}$. Hence its trace is zero. However its trace is the image of the symmetric invariant $\mathrm{Tr}(X^{2n})$ by the transgression operator, then it cannot be zero by the Hopf-Koszul-Samelson theorem (see for instance [10]). We have a contradiction. In consequence, $\mathscr{A}_{4n}$ is not identically zero. □

Remark 5.6. *From the equation (4.3), we have $\mathscr{A}_k|_{\mathfrak{g}^k} = 0$ if $k \geq 4n+2$. Also we just checked that $\mathscr{A}_{4n}|_{\mathfrak{g}_{\bar{0}}^{4n-1}\times\mathfrak{g}_{\bar{1}}}$ $\neq 0$. So the index obtained in Theorem 5.1 is the best possible, if one considers only even indices, a technical but justified assumption (see* [11]*).*

As for $\mathscr{A}_{4n+1}|_{\mathfrak{g}^{4n+1}}$, it does not vanish if $n = 1, 2$ and 3. To verify this, we used once again the realization of $\mathfrak{osp}(1,2n)$ into the Weyl algebra A_n (see 2.1). More precisely, set $y_i = [p_i,q_i] = p_iq_i + q_ip_i$ and $x_j = [p_j,q_{j+1}]$ $\in \mathsf{A}_n$ (for $1 \leq i \leq n$ and $1 \leq j \leq n-1$). We have:

- *$n = 1$: $\mathscr{A}_5(p_1,q_1,y_1,p_1,q_1)(p_1) = -2^6 p_1$ (this can be easily checked by hand).*
- *$n = 2$: $\mathscr{A}_9(p_1,q_1,y_1,p_1,q_1,x_1,y_2,p_2,q_2)(p_2) = 2^{11} p_1$*
- *$n = 3$: $\mathscr{A}_{13}(p_1,q_1,y_1,p_1,q_1,x_1,y_2,p_2,q_2,x_2,y_3,p_3,q_3)(p_3) = -2^{15}\ 3\ p_1$*

The last two computations were performed using `Maple`*, the case $n = 3$ taking several hours to be finished. The general case is still to be done, nevertheless we conjecture that:*

$$\mathscr{A}_{4n+1}(p_1,q_1,y_1,p_1,q_1,x_1,y_2,p_2,q_2,\dots,x_{n-1},y_n,p_n,q_n)(p_n) = (-1)^n n!\ 2^{4n+2}\ p_1$$

REFERENCES

[1] Amitsur, A. S. and Levitzki J., Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1** (1950), 449 – 463.

[2] Benanti F. , Demmel J. , Drensky V. , Koev P., Computational approach to polynomial identities of matrices - a survey, *Polynomial identities and combinatorial methods, A. Giambruno, ed.*, Dekker, 2003.

[3] Cartan, H., La transgression dans un groupe de Lie et dans un espace fibré principal, *Coll. Topologie, C. B. R. M. Bruxelles*, (1950), 57 – 71.

[4] Chevalley, C., The Betti numbers of the exceptional Lie groups, *Proc. Intern. Congress of Math.* **II** (1950), 21 – 24.

[5] Djokovic, D. Z. and Hochschild, G., Semi-simplicity of 2-graded Lie algebras, *Illinois J. of Math.* **20** (1976), 134 – 143.

[6] Fuks, D. B. and Leites D. H., Cohomology of Lie superalgebras, *C. R. Acad. Bulg. Sci.* **37** (1984), 1595 – 1596.

[7] Gié, P-A. , Pinczon G. , Ushirobira R., Back to the Amitsur-Levitzki theorem: a super version for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2n)$, *Letters in Mathematical Physics* **66** (2003), 141 –155.

[8] Jacobson, N., PI-Algebras, *Lecture Notes in Mathematics* **441**, Springer-Verlag, Berlin, 1975.

[9] Kantor, I. and Trishin, I. , The algebra of polynomial invariants of the adjoint representation of the Lie superalgebra $\mathfrak{gl}(m,n)$, *Comm. Algebra* **25** (1997), no. 7, 2039–2070.

[10] Kostant, B., Clifford analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \mathrm{End}\,V_\rho \otimes C(P)$ and the $\mathfrak{g}$-module structure of $\bigwedge\mathfrak{g}$, *Adv. in Math.* **125** (1997), 275 – 350.

[11] Kostant, B., A Lie algebra generalization of the Amitsur-Levitzki theorem, *Adv. in Math.* **40** (1981), 155 – 175.

[12] Kostant, B., Lie groups representations on polynomial rings, *Amer. J. Math.* **85** (1963), 327 – 404.

[13] Kostant, B., A theorem of Frobenius, a theorem of Amitsur-Levitzki and cohomology theory, *J. Math. and Mech.* **7** (1958), 237 – 264.

[14] Koszul, J. L., Homologie et cohomologie des algèbres de Lie, *Bull. Soc. Math. Fr.* **78** (1950), 65 – 127.
[15] Leites, D. H., Cohomology of Lie superalgebras, *Funct. Anal. Appl.* **9** (1975), 75 – 76.
[16] Musson, I. M., Enveloping algebras of Lie super algebras, a survey, *Contemporary Mathematics* **124** (1992), 141 – 149.
[17] Musson, I. M., On the center of the enveloping algebra of a classical simple Lie superalgebras, *J. of Algebra* **193** (1997), 75 – 101.
[18] Pinczon, G., The enveloping algebra of the Lie superalgebra $\mathfrak{osp}(1,2)$, *J. of Algebra* **132** (1990), 219 – 242.
[19] Razmyslov, Yu. P., Identities of algebras and their representations. Translated from the 1989 Russian original, *Translations of Math. Monographs*, **138**, xiv+318 pp. American Mathematical Society, Providence, RI, 1994.
[20] Rosset, S., A new proof of the Amitsur-Levitzki theorem, *Israel J. Math.* **23** (1976), 187 – 188.
[21] Rowen, L. H., Standard polynomials in matrix algebras, *Trans. Amer. Math.* **190** (1974), 253 – 284.
[22] Rowen, L. H., Polynomials identities in ring theory, *Academic Press*, New York, 1980.
[23] Sergeev, A. N., The invariant polynomials on simple Lie superalgebras, *Representation Theory* **3** (1999), 250 – 280.
[24] Swan, R. G., An application of graph theory to algebra, *Proc. Amer. Math. Soc.* **14** (1963), 367 – 373. Correction at *ibid.* **21** (1969) 397 – 380.

INSTITUT DE MATHÉMATIQUES DE BOURGOGNE, UNIVERSITÉ DE BOURGOGNE, B.P. 47870, F-21078 DIJON CEDEX, FRANCE
E-mail address: `pagie, gpinczon, rosane@u-bourgogne.fr`